



TITLE:

# Theory and Numerical Analysis of the Boltzmann Equation : Theory and Analysis of Rarefied Gas Flows Part I (Chapter 1-3)

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CITATION:

Sone, Yoshio. Theory and Numerical Analysis of the Boltzmann Equation : Theory and Analysis of Rarefied Gas Flows Part I (Chapter 1-3). 1998

ISSUE DATE:

1998-08-17

URL:

<http://hdl.handle.net/2433/65065>

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**Lecture Notes**  
**Department of Aeronautics and**  
**Astronautics**  
**Graduate School of Engineering**  
**Kyoto University**

**Theory and Numerical Analysis of the**  
**Boltzmann Equation**  
Theory and Analysis of Rarefied Gas Flows  
**Part I (Chapter 1 – 3)**

**Yoshio Sone**

17 August 1998



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# Preface

The present lecture notes is devoted to analytical and numerical studies of the behavior of solutions of the Boltzmann equation for those who know what the Boltzmann equation is. The materials are chosen according to their physical importance, and thus the notes also provides the up-to-date fundamental information of molecular gas dynamics.

Chapters 1, 2, and 3 are prepared by Y. Sone, and Chapters 4 and 5 will be prepared by K. Aoki, and Appendix will be prepared by the cooperation of the two.

Considerable part of the work was done while Y. Sone was staying at University of Parma as a visiting professor of Italian CNR program. He expresses his cordial thanks to Professor and Mrs. Giampiero Spiga for their kind hospitality during the stay. Y. Sone also expresses his thanks to Drs. S. Takata and T. Doi for their help in eliminating flaws in the manuscript and Dr. H. Sugimoto and Messrs. A. Okuda and M. Handa for their assistance in preparing figures.

*Yoshio Sone*  
Kyoto, 1998

## Notes:

Some references are supplemented.

*Yoshio Sone*  
Kyoto, 2000

The original plan of the lecture notes is replaced by the publication of the following two books:

Y. Sone, *Kinetic Theory and Fluid Dynamics* (Birkhäuser, Boston, 2002),

Y. Sone, *Molecular Gas Dynamics* (Birkhäuser, Boston, 2007).

They contain much more materials with detailed discussions and up-to-date results as well as the explanation of the Boltzmann equation.

No revision of the lecture notes is made on the occasion of its registration in Kyoto University Research Information Repository (KURIR), which is accessed at (<http://repository.kuil.kyoto-u.ac.jp>).

*Yoshio Sone*  
Kyoto, 2008



# Chapter 1

## Boltzmann Equation

In this chapter we summarize the preliminary information on the Boltzmann equation (the definitions, notations, formulas, etc.) for the convenience of the discussions in the following chapters. Further information is found in Boltzmann [1964], Grad [1958], Kogan [1969], Cercignani [1987], and Sone & Aoki [1994].

### 1.1 Velocity distribution function and macroscopic variables

Let  $X_i$  (or  $\mathbf{X}$ ) be the Cartesian coordinates of our physical space, and  $\xi_i$  (or  $\boldsymbol{\xi}$ ) the molecular velocity. Let the number  $dN$  of molecules in the six-dimensional volume element  $dX_1 dX_2 dX_3 d\xi_1 d\xi_2 d\xi_3$  ( $d\mathbf{X} d\boldsymbol{\xi}$  for short) be expressed as

$$dN = m^{-1} f(\mathbf{X}, \boldsymbol{\xi}, t) d\mathbf{X} d\boldsymbol{\xi}, \quad (1.1)$$

where  $m$  is the mass of a molecule and  $t$  is the time. Then,  $f$  or  $m^{-1}f$ , which is a function of the seven variables  $\mathbf{X}$ ,  $\boldsymbol{\xi}$ , and  $t$ , is called the velocity distribution function of the gas molecules.

The macroscopic variables: the density  $\rho$  of the gas, the flow velocity  $v_i$ , the temperature  $T$ , the pressure  $p$ , the specific internal energy  $e$ , the stress tensor  $p_{ij}$ , and the heat-flow vector  $q_i$ , are defined by the following moments of  $f$ :

$$\rho = \int f d\boldsymbol{\xi}, \quad (1.2)$$

$$v_i = \frac{1}{\rho} \int \xi_i f d\boldsymbol{\xi}, \quad (1.3)$$

$$3RT = \frac{1}{\rho} \int (\xi_i - v_i)^2 f d\boldsymbol{\xi}, \quad (1.4)$$

$$p = \frac{1}{3} \int (\xi_i - v_i)^2 f d\boldsymbol{\xi} = R\rho T, \quad (1.5)$$



$$e = \frac{1}{\rho} \int \frac{1}{2} (\xi_i - v_i)^2 f d\boldsymbol{\xi} = \frac{3}{2} RT, \quad (1.6)$$

$$p_{ij} = \int (\xi_i - v_i)(\xi_j - v_j) f d\boldsymbol{\xi}, \quad (1.7)$$

$$q_i = \int \frac{1}{2} (\xi_i - v_i)(\xi_j - v_j)^2 f d\boldsymbol{\xi}, \quad (1.8)$$

where  $R$  is the specific gas constant, i.e., the Boltzmann constant  $k_B$  ( $= 1.380658 \times 10^{-23} \text{J} \cdot \text{K}^{-1}$ ) divided by  $m$ , and the three dimensional integration with respect to  $\boldsymbol{\xi}$  is, hereafter, carried out over the whole space of  $\boldsymbol{\xi}$  unless otherwise stated.

The mass  $-M$ , momentum  $-P_i$ , and energy  $-EF$  transferred from the gas to the boundary per its unit area and per unit time are given by

$$\begin{aligned} M &= \int (\xi_j - v_{wj}) n_j f(\mathbf{X}, \boldsymbol{\xi}, t) d\boldsymbol{\xi} \\ &= n_j \rho (v_j - v_{wj}), \end{aligned} \quad (1.9)$$

$$\begin{aligned} P_i &= \int \xi_i (\xi_j - v_{wj}) n_j f(\mathbf{X}, \boldsymbol{\xi}, t) d\boldsymbol{\xi} \\ &= n_j [p_{ij} + \rho v_i (v_j - v_{wj})], \end{aligned} \quad (1.10)$$

$$\begin{aligned} EF &= \int \frac{1}{2} \xi_i^2 (\xi_j - v_{wj}) n_j f(\mathbf{X}, \boldsymbol{\xi}, t) d\boldsymbol{\xi} \\ &= n_j \left[ q_j + p_{ij} v_i + \rho \left( e + \frac{1}{2} v_i^2 \right) (v_j - v_{wj}) \right], \end{aligned} \quad (1.11)$$

where  $v_{wi}$  is the velocity of the boundary and  $n_i$  is the unit normal vector to the boundary, pointed to the gas.

## 1.2 Boltzmann equation

The behavior of  $f$  is determined by the following Boltzmann equation:

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial X_i} + \frac{\partial F_i f}{\partial \xi_i} = J(f, f), \quad (1.12)$$

where

$$J(f, f) = \frac{1}{m} \int_{\text{all } \alpha_i, \text{ all } \xi_{i*}} (f' f'_* - f f_*) B(|\alpha_j V_j|, V) d\Omega(\boldsymbol{\alpha}) d\boldsymbol{\xi}_*, \quad (1.13)$$

$$\left. \begin{aligned} f &= f(X_i, \xi_i, t), & f_* &= f(X_i, \xi_{i*}, t), \\ f' &= f(X_i, \xi'_i, t), & f'_* &= f(X_i, \xi'_{i*}, t), \\ \xi'_i &= \xi_i + \alpha_i \alpha_j V_j, & \xi'_{i*} &= \xi_{i*} - \alpha_i \alpha_j V_j, \\ V_i &= \xi_{i*} - \xi_i, & V &= (V_i^2)^{1/2}, \end{aligned} \right\} \quad (1.14)$$

and  $mF_i$  is the external force, such as the gravitational force, on a molecule,  $\alpha_i$  (or  $\boldsymbol{\alpha}$ ) is a unit vector, expressing the variation of the direction of the molecular

velocity owing to a molecular collision,  $d\Omega(\boldsymbol{\alpha})$  is the solid-angle element in the direction of  $\alpha_i$ , and  $B(|\alpha_i V_i|, V)$  is a nonnegative function of  $|\alpha_i V_i|$  and  $V$ , whose functional form is determined by the intermolecular force (e.g., for a gas consisting of hard-sphere molecules with diameter  $d_m$ ,  $B = d_m^2 |V_i \alpha_i|/2$ ). The integrations with respect to  $\xi_{i*}$  and  $\alpha_i$  are carried out over the whole space of  $\xi_{i*}$  and over the whole direction of  $\alpha_i$  (the whole spherical surface) respectively.

### 1.3 Conservation equation

Multiplying the Boltzmann equation (1.12) by 1,  $\xi_i$ , or  $\xi_i^2$ , and integrating the result over the whole space of  $\boldsymbol{\xi}$ , we obtain the following conservation equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial X_i}(\rho v_i) = 0, \quad (1.15)$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial X_j}(\rho v_i v_j + p_{ij}) = \rho F_i, \quad (1.16)$$

$$\frac{\partial}{\partial t} \left[ \rho \left( e + \frac{1}{2} v_i^2 \right) \right] + \frac{\partial}{\partial X_j} \left[ \rho v_j \left( e + \frac{1}{2} v_i^2 \right) + v_i p_{ji} + q_j \right] = \rho v_j F_j, \quad (1.17)$$

where the force  $F_i$  is assumed to be independent of the molecular velocity  $\boldsymbol{\xi}$ . The collision term vanishes on integration. These are the conservation equations of mass, momentum, and energy in the continuum theory, where  $p_{ij}$  and  $q_i$  are assumed to be in appropriate forms to close the system (1.15)–(1.17).

### 1.4 Maxwell distribution (Equilibrium distribution)

The solution  $f_e$  of the Boltzmann equation (1.12) with  $F_i = 0$  for an equilibrium state ( $\partial f / \partial t = \partial f / \partial X_i = 0$ ) is given by the following Maxwellian distribution with five constant parameters  $\rho$ ,  $v_i$ , and  $T$ :

$$f_e = \frac{\rho}{(2\pi RT)^{3/2}} \exp \left( -\frac{(\xi_i - v_i)^2}{2RT} \right). \quad (1.18)$$

If  $\rho$ ,  $v_i$ , and  $T$  depend on  $X_i$  or  $t$ , the distribution does not satisfy Eq. (1.12) with  $F_i = 0$  except for some special cases (see Grad [1949]). The distribution is called local Maxwellian.

### 1.5 Mean free path

The collision frequency  $\nu_c(\boldsymbol{\xi})$  of a molecule with velocity  $\xi_i$  per unit time is

$$\nu_c = m^{-1} \int f(\boldsymbol{\xi}_*) B d\Omega(\boldsymbol{\alpha}) d\boldsymbol{\xi}_*. \quad (1.19)$$

Its average over all the molecules (the mean collision frequency  $\bar{\nu}_c$ ) is

$$\bar{\nu}_c = \rho^{-1} m^{-1} \int f(\boldsymbol{\xi}) f(\boldsymbol{\xi}_*) B d\Omega d\boldsymbol{\xi}_* d\boldsymbol{\xi}. \quad (1.20)$$

The inverses of  $\nu_c(\boldsymbol{\xi})$  and  $\bar{\nu}_c$  are called, respectively, the free time  $\tau_c(\boldsymbol{\xi})$  [=  $1/\nu_c(\boldsymbol{\xi})$ ] of the molecule  $\boldsymbol{\xi}$  and the mean free time  $\bar{\tau}_c$  of the gas molecules. The mean free time multiplied by the average speed  $\bar{c}$  [=  $\rho^{-1} \int (\xi_i^2)^{1/2} f d\boldsymbol{\xi}$ ] of the gas molecules is called the mean free path  $\ell$  of the gas molecules:

$$\ell = \bar{c} \bar{\tau}_c. \quad (1.21)$$

If the velocity distribution function is a Maxwellian with  $v_i = 0$ , then  $\ell$  is given by

$$\ell = [\sqrt{2\pi} d_m^2 (\rho/m)]^{-1}, \quad (1.22)$$

where  $d_m$  is the radius of the influence range of the intermolecular force. [This is the diameter of a hard-sphere molecule.]

## 1.6 Boundary condition

### 1.6.1 Simple boundary

On a boundary or wall where there is no mass flux across it (a simple boundary, for short), the following condition (the Maxwell-type condition) is widely used.

$$f(X_i, \xi_i, t) = (1 - \alpha) f(X_i, \xi_i - 2(\xi_j - v_{wj}) n_j n_i, t) + \frac{\alpha \bar{\rho}_w}{(2\pi R T_w)^{3/2}} \exp\left(-\frac{(\xi_j - v_{wj})^2}{2 R T_w}\right), \quad [(\xi_j - v_{wj}) n_j > 0], \quad (1.23a)$$

$$\bar{\rho}_w = - \left( \frac{2\pi}{R T_w} \right)^{1/2} \int_{(\xi_j - v_{wj}) n_j < 0} (\xi_j - v_{wj}) n_j f(X_i, \xi_i, t) d\boldsymbol{\xi}, \quad (1.23b)$$

where  $T_w$  and  $v_{wi}$  are, respectively, the temperature and velocity of the boundary,  $n_i$  is the unit normal vector to the boundary, pointed to the gas, and  $\alpha$  ( $0 \leq \alpha \leq 1$ ) is the accommodation coefficient.<sup>1</sup> These quantities depend on the position of the boundary. The case  $\alpha = 1$  is called the diffuse reflection, and  $\alpha = 0$  the specular reflection.

More generally, the boundary condition is expressed in terms of a scattering kernel as

$$f(X_i, \xi_i, t) = \int_{(\tilde{\xi}_j - v_{wj}) n_j < 0} K_B(\xi_i, \tilde{\xi}_i, X_i, t) f(X_i, \tilde{\xi}_i, t) d\tilde{\boldsymbol{\xi}}, \quad [(\xi_j - v_{wj}) n_j > 0]. \quad (1.24)$$

<sup>1</sup>When the condition of the boundary is specified, it is generally rather loosely mentioned that the temperature or velocity of the body or the condensed phase under consideration is so and so. This means that the surface temperature or velocity of the body or the condensed phase is so and so; then the temperature or velocity inside of it is not important. Hereafter we do not repeat this type of note.

The kernel  $K_B(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}, \mathbf{X}, t)$  is required to satisfy the following conditions:

$$(i) \quad K_B(\xi_i, \tilde{\xi}_i) \geq 0, \quad [(\xi_j - v_{wj})n_j > 0, (\tilde{\xi}_j - v_{wj})n_j < 0], \quad (1.25a)$$

$$(ii) \quad - \int_{(\xi_j - v_{wj})n_j > 0} \frac{(\xi_k - v_{wk})n_k}{(\tilde{\xi}_j - v_{wj})n_j} K_B(\xi_i, \tilde{\xi}_i) d\boldsymbol{\xi} = 1, \quad [(\tilde{\xi}_j - v_{wj})n_j < 0], \quad (1.25b)$$

$$(iii) \quad f_B(\xi_i) = \int_{(\tilde{\xi}_j - v_{wj})n_j < 0} K_B(\xi_i, \tilde{\xi}_i) f_B(\tilde{\xi}_i) d\tilde{\boldsymbol{\xi}}, \quad [(\xi_j - v_{wj})n_j > 0], \quad (1.25c)$$

where

$$f_B(\xi_i) = \frac{\rho}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_i - v_{wi})^2}{2RT_w}\right),$$

with  $\rho$  being arbitrary, and the other Maxwellians do not satisfy the relation (1.25c). This uniqueness condition excludes the specular reflection.

### 1.6.2 Interface

On the interface of a gas and its condensed phase, the following condition is often used:

$$\begin{aligned} f = & \frac{\alpha_c \rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_j - v_{wj})^2}{2RT_w}\right) \\ & + (1 - \alpha_c) \left[ (1 - \alpha) f(X_i, \xi_i - 2(\xi_j - v_{wj})n_j n_i, t) \right. \\ & \left. + \frac{\alpha \bar{\rho}_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_j - v_{wj})^2}{2RT_w}\right) \right], \quad [(\xi_j - v_{wj})n_j > 0], \end{aligned} \quad (1.26a)$$

$$\bar{\rho}_w = - \left( \frac{2\pi}{RT_w} \right)^{1/2} \int_{(\xi_j - v_{wj})n_j < 0} (\xi_j - v_{wj})n_j f(X_i, \xi_i, t) d\boldsymbol{\xi}, \quad (1.26b)$$

where  $\rho_w$  is the saturation gas density at temperature  $T_w$ ,  $\alpha_c$  ( $0 \leq \alpha_c \leq 1$ ) is the condensation coefficient, and  $\alpha$  ( $0 \leq \alpha \leq 1$ ) is the accommodation coefficient. Especially the case  $\alpha_c = 1$  (the complete-condensation condition) is widely used. The experimental discussion is found in Taken, Mischke, Korving & Beenakker [1984]. Hereafter, the saturation gas pressure  $p_w$  ( $= R\rho_w T_w$ ) at temperature  $T_w$  is more often used instead of  $\rho_w$ .

More generally, the condition is expressed in terms of a scattering kernel as

$$\begin{aligned} f(X_i, \xi_i, t) = & g_B(X_i, \xi_i, t) + \int_{(\tilde{\xi}_j - v_{wj})n_j < 0} K_I(\xi_i, \tilde{\xi}_i, X_i, t) f(X_i, \tilde{\xi}_i, t) d\tilde{\boldsymbol{\xi}}, \\ & [(\xi_j - v_{wj})n_j > 0], \end{aligned} \quad (1.27)$$

where  $g_B$ , independent of  $f$ , corresponds to the term including  $\rho_w$  in Eq. (1.26a). The  $g_B$  and  $K_I(\boldsymbol{\xi}, \tilde{\boldsymbol{\xi}}, \mathbf{X}, t)$  is required to satisfy the following conditions:

$$(i) g_B(X_i, \xi_i, t) \geq 0, \quad [(\xi_j - v_{wj})n_j > 0], \quad (1.28a)$$

$$(ii) K_I(\xi_i, \tilde{\xi}_i) \geq 0, \quad [(\xi_j - v_{wj})n_j > 0, (\tilde{\xi}_j - v_{wj})n_j < 0], \quad (1.28b)$$

$$(iii) f_w(\xi_i) = g_B(X_i, \xi_i, t) + \int_{(\tilde{\xi}_j - v_{wj})n_j < 0} K_I(\xi_i, \tilde{\xi}_i) f_w(\tilde{\xi}_i) d\tilde{\boldsymbol{\xi}}, \quad [(\xi_j - v_{wj})n_j > 0], \quad (1.28c)$$

where

$$f_w = \frac{\rho_w}{(2\pi RT_w)^{3/2}} \exp\left(-\frac{(\xi_i - v_{wi})^2}{2RT_w}\right),$$

and the other Maxwellians do not satisfy the relation (1.28c).

## 1.7 H theorem

Consider the following functional of the velocity distribution function  $f$  :

$$H(X_i, t) = \int f \ln(f/c_0) d\boldsymbol{\xi}. \quad (1.29)$$

The time variation of  $H$  is given by

$$\frac{\partial H}{\partial t} + \frac{\partial H_i}{\partial X_i} = G, \quad (1.30)$$

where

$$H_i = \int \xi_i f \ln(f/c_0) d\boldsymbol{\xi}, \quad (1.31)$$

$$G = -\frac{1}{4m} \int (ff_*' - ff_*') \ln\left(\frac{ff_*'}{ff_*}\right) B d\Omega d\boldsymbol{\xi}_* d\boldsymbol{\xi} \leq 0. \quad (1.32)$$

The equality in the last relation holds when and only when  $f$  is a (local) Maxwellian. Then the time variation of the integral  $\bar{H}$  of  $H$  over a domain  $D$  bounded by  $\partial D$ , which may be a moving boundary, is given by

$$\frac{d\bar{H}}{dt} - \int_{\partial D} (H_i - H v_{wi}) n_i dS = \int_D G d\mathbf{X} \leq 0, \quad (1.33)$$

where  $v_{wi}$  is the velocity of the boundary and  $n_i$  is the unit normal vector to the boundary, pointed to the gas.

From these equations, we have

(i) If the state, or  $f$ , is spatially uniform, then  $H$  never increases.

(ii) If  $(H_i - H v_{wi}) n_i = 0$  on the boundary  $\partial D$ , then  $\bar{H}$  never increase.

In both cases,  $H$  or  $\bar{H}$  remains constant only when  $f$  is a (local) Maxwellian. These are the Boltzmann H theorem, which shows the time evolution of a solution of the Boltzmann equation has a direction.

If the boundary is a simple boundary, the second statement is made more explicit in Darrozes & Guiraud [1966] by the estimate of the boundary contribution:

$$\int_{\partial D} (H_i - H v_{wi}) n_i dS \leq - \int_{\partial D} \frac{n_i [q_i + p_{ij}(v_j - v_{wj})]}{RT_w} dS. \quad (1.34)$$

Then,

$$\frac{d\bar{H}}{dt} \leq - \int_{\partial D} \frac{n_i [q_i + p_{ij}(v_j - v_{wj})]}{RT_w} dS, \quad (1.35)$$

where the equality holds only when  $f$  is the Maxwellian that is a solution of the Boltzmann equation with the boundary condition. In this case the right hand side vanishes. From this relation, we have

(ii') If there is no heat flow that flows into the boundary on each point of a simple boundary,<sup>2</sup> then  $\bar{H}$  never increases.

## 1.8 Model equation

The following model equation [Boltzmann–Krook–Welanders, BKW, or BGK equation (Bhatnagar, Gross & Krook [1954], Welanders [1954], Kogan [1958], Sone & Aoki [1994])], where the collision term in Eq. (1.12) is simplified, is widely used in analyses of rarefied gas flows:

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial X_i} + \frac{\partial F_i f}{\partial \xi_i} = A_c \rho (f_e - f), \quad (1.36)$$

where  $A_c$  is a constant ( $A_c \rho$  gives the collision frequency independent of  $\xi$ ), and  $f_e$  is the local Maxwellian with  $\rho$ ,  $v_i$ , and  $T$  derived from  $f$  [Eqs.(1.2)–(1.4)] as its five parameters.

In the above simplification of the collision integral, the velocity distribution function  $f$  of the gas molecules is not assumed to be close to the local Maxwellian  $f_e$ , but the difference between the velocity distribution function of the molecules just after collision during the mean free time  $(A_c \rho)^{-1}$  and the local Maxwellian  $f_e$  is assumed to be small compared with the difference of the velocity distribution function  $f$  and the local Maxwellian  $f_e$  (thus,  $f$  may differ considerably from  $f_e$ ).

For the BKW equation, the same conservation equations as in Section 1.3, where the collision term has dropped off by integration (this is an important property of the collision term), are derived, and the H theorem (Section 1.7), an important property of the Boltzmann equation, holds.

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<sup>2</sup>On a simple boundary, the energy transferred to the boundary from the gas is  $-n_i(q_i + p_{ij}v_j)$  [Eq. (1.11)], and the work done on the boundary by the gas is  $-n_i p_{ij}v_{wj}$ . Thus, their difference  $-n_i[q_i + p_{ij}(v_j - v_{wj})]$  is the heat flow into the boundary.

## 1.9 Nondimensional expressions

The nondimensional variables and equations, which are often used in the following chapters, are listed here. The variables are chosen in such a way that they express a perturbation from a Maxwellian distribution with  $v_i = 0$ .

Let  $L$ ,  $t_0$ ,  $T_0$ , and  $p_0$  be the reference length, time, temperature, and pressure, and put

$$\rho_0 = p_0(RT_0)^{-1}. \quad (1.37)$$

As the reference state, we take the Maxwellian distribution with  $v_i = 0$ ,  $p = p_0$ , and  $T = T_0$ :

$$f_0 = \frac{\rho_0}{(2\pi RT_0)^{3/2}} \exp\left(-\frac{\xi_i^2}{2RT_0}\right). \quad (1.38)$$

Then, the nondimensional variables are defined as follows:

$$\left. \begin{aligned} x_i &= X_i/L, & \bar{t} &= t/t_0, & \zeta_i &= \xi_i/(2RT_0)^{1/2}, \\ \phi &= f/f_0 - 1, & \omega &= \rho/\rho_0 - 1, & u_i &= v_i/(2RT_0)^{1/2}, \\ \tau &= T/T_0 - 1, & P &= p/p_0 - 1, & P_{ij} &= p_{ij}/p_0 - \delta_{ij}, \\ Q_i &= q_i/p_0(2RT_0)^{1/2}, & u_{wi} &= v_{wi}/(2RT_0)^{1/2}, & \tau_w &= T_w/T_0 - 1, \\ \omega_w &= \rho_w/\rho_0 - 1, & P_w &= p_w/p_0 - 1, & (P_w &= \omega_w + \tau_w + \omega_w\tau_w). \end{aligned} \right\} \quad (1.39)$$

With the notation  $E(\zeta)$

$$E(\zeta) = \pi^{-3/2} \exp(-\zeta_i^2), \quad (1.40)$$

the reference distribution  $f_0$  is expressed as

$$f_0 = \rho_0(2RT_0)^{-3/2} E(\zeta). \quad (1.41)$$

The nondimensional form of the Boltzmann equation with  $F_i = 0$  is given as

$$Sh \frac{\partial \phi}{\partial \bar{t}} + \zeta_i \frac{\partial \phi}{\partial x_i} = \frac{1}{k} [\mathcal{L}(\phi) + \hat{J}(\phi, \phi)], \quad (1.42)$$

$$\mathcal{L}(\phi) = \int E_*(\phi' + \phi'_* - \phi - \phi_*) \hat{B} d\Omega(\alpha) d\zeta_*, \quad (1.43)$$

$$\hat{J}(\phi, \psi) = \frac{1}{2} \int E_*(\phi' \psi'_* + \phi'_* \psi' - \phi \psi_* - \phi_* \psi) \hat{B} d\Omega(\alpha) d\zeta_*, \quad (1.44)$$

$$\left. \begin{aligned} Sh &= Lt_0^{-1}(2RT_0)^{-1/2}, & \hat{B} &= BB_0^{-1}, \\ k &= (2RT_0)^{1/2}(\rho_0 B_0 L)^{-1} m = (\sqrt{\pi}/2)\ell_0 L^{-1} = (\sqrt{\pi}/2)Kn, \\ B_0 &= \int EE_* B d\Omega(\alpha) d\zeta d\zeta_*, & d\zeta &= d\zeta_1 d\zeta_2 d\zeta_3, & d\zeta_* &= d\zeta_{1*} d\zeta_{2*} d\zeta_{3*} \end{aligned} \right\} \quad (1.45)$$

where the variable  $\zeta_i$  in  $\phi_*$ ,  $\phi'_*$ , and  $\phi'_*$  (and also for other quantities) are replaced by  $\zeta_{i*}$ ,  $\zeta'_i$ , and  $\zeta'_{i*}$  respectively:

$$\zeta'_i = \zeta_i + \alpha_i \alpha_j (\zeta_{j*} - \zeta_j), \quad \zeta'_{i*} = \zeta_{i*} - \alpha_i \alpha_j (\zeta_{j*} - \zeta_j). \quad (1.46)$$

The  $Sh$  is the Strouhal number,  $\rho_0 B_0 m^{-1}$  is the mean collision frequency of the reference state,  $\ell_0$  is the corresponding mean free path. The Knudsen number  $Kn$  is an important parameter characterizing the rate of rarefaction of the gas. The nondimensionalized form  $\widehat{B}$  of  $B$  is a function of  $(\zeta_{i*} - \zeta_i)^2$ ,  $|\alpha_i(\zeta_{i*} - \zeta_i)|$ , and  $U_0/k_B T_0$ , where  $U_0$  is the characteristic size of the intermolecular potential. For a hard-sphere molecular gas,

$$\widehat{B} = |\alpha_i(\zeta_{i*} - \zeta_i)|/4(2\pi)^{1/2}. \quad (1.47)$$

The relations of the nondimensional macroscopic variables and the nondimensional velocity distribution function  $\phi$  are

$$\left. \begin{aligned} \omega &= \int \phi E d\zeta, & (1 + \omega)u_i &= \int \zeta_i \phi E d\zeta, \\ \frac{3}{2}(1 + \omega)\tau &= \int \left( \zeta_i^2 - \frac{3}{2} \right) \phi E d\zeta - (1 + \omega)u_i^2, \\ P &= \omega + \tau + \omega\tau, & P_{ij} &= 2 \int \zeta_i \zeta_j \phi E d\zeta - 2(1 + \omega)u_i u_j, \\ Q_i &= \int \zeta_i \zeta_j^2 \phi E d\zeta - \frac{5}{2}u_i - u_j P_{ij} - \frac{3}{2}P u_i - (1 + \omega)u_i u_j^2. \end{aligned} \right\} \quad (1.48)$$

The linear part  $\mathcal{L}(\phi)$  of the collision integral, called linearized collision integral, for hard-sphere molecules is expressed in terms of collision kernel in Grad [1963] as follows:

$$\mathcal{L}(\phi) = \mathcal{L}_1(\phi) - \mathcal{L}_2(\phi) - \nu(\zeta)\phi, \quad (1.49)$$

where

$$\mathcal{L}_1(\phi) = \frac{1}{\sqrt{2\pi}} \int \frac{1}{|\zeta_j - \tilde{\zeta}_j|} \exp \left( -\tilde{\zeta}_j^2 + \frac{(\epsilon_{klm} \zeta_l \tilde{\zeta}_m)^2}{(\zeta_j - \tilde{\zeta}_j)^2} \right) \phi(x_i, \tilde{\zeta}_i, \bar{t}) d\tilde{\zeta}, \quad (1.50a)$$

$$\mathcal{L}_2(\phi) = \frac{1}{2^{3/2}\pi} \int |\zeta_j - \tilde{\zeta}_j| \exp(-\tilde{\zeta}_k^2) \phi(x_i, \tilde{\zeta}_i, \bar{t}) d\tilde{\zeta}, \quad (1.50b)$$

$$\nu(\zeta) = \frac{1}{2^{3/2}} \left[ \exp(-\zeta^2) + \left( 2\zeta + \frac{1}{\zeta} \right) \int_0^\zeta \exp(-\tilde{\zeta}^2) d\tilde{\zeta} \right], \quad (1.50c)$$

where  $\epsilon_{ijk}$  is Eddington's epsilon and  $\zeta = (\zeta_i^2)^{1/2}$ .

## 1.10 The linearized Boltzmann equation

In analyzing the behavior of a gas which deviates only slightly from an equilibrium state at rest, the linearized version of the Boltzmann equation, which is obtained by neglecting the nonlinear terms of  $\phi$  in Eq. (1.42), is widely used. It is given as

$$Sh \frac{\partial \phi}{\partial t} + \zeta_i \frac{\partial \phi}{\partial x_i} = \frac{1}{k} \mathcal{L}(\phi). \quad (1.51)$$



The linearized version of Eq. (1.48) is

$$\left. \begin{aligned} \omega &= \int \phi E d\zeta, & u_i &= \int \zeta_i \phi E d\zeta, & \frac{3}{2}\tau &= \int \left( \zeta_i^2 - \frac{3}{2} \right) \phi E d\zeta, \\ P &= \omega + \tau, & P_{ij} &= 2 \int \zeta_i \zeta_j \phi E d\zeta, & Q_i &= \int \zeta_i \zeta_j^2 \phi E d\zeta - \frac{5}{2} u_i. \end{aligned} \right\} \quad (1.52)$$

The linearized expression  $\phi_e$  for  $\phi$  corresponding to a local Maxwellian is

$$\phi_e = \omega + 2\zeta_i u_i + \left( \zeta_i^2 - \frac{3}{2} \right) \tau. \quad (1.53)$$

The linearized BKW equation is

$$Sh \frac{\partial \phi}{\partial t} + \zeta_i \frac{\partial \phi}{\partial x_i} = \frac{1}{k} \left[ -\phi + \omega + 2\zeta_i u_i + \left( \zeta_i^2 - \frac{3}{2} \right) \tau \right], \quad (1.54)$$

where

$$k = (2RT_0)^{1/2} (\rho_0 A_c L)^{-1} = (\sqrt{\pi}/2) \ell_0 L^{-1}. \quad (1.55)$$

The linearized boundary condition corresponding to Eq. (1.23a) with Eq. (1.23b) on a simple boundary is

$$\begin{aligned} \phi(x_i, \zeta_i, \bar{t}) &= (1 - \alpha) \left[ \phi(x_i, \zeta_i - 2(\zeta_j - u_{wj})n_j n_i, \bar{t}) + 4\zeta_j u_{wk} n_j n_k \right] \\ &+ \alpha \left[ \sigma_w + 2\zeta_j u_{wj} + \left( \zeta_j^2 - \frac{3}{2} \right) \tau_w \right], \quad [(\zeta_i - u_{wi})n_i > 0], \end{aligned} \quad (1.56a)$$

$$\sigma_w = \sqrt{\pi} u_{wj} n_j - \frac{1}{2} \tau_w - 2\sqrt{\pi} \int_{(\zeta_k - u_{wk})n_k < 0} (\zeta_j - u_{wj}) n_j \phi E d\zeta. \quad (1.56b)$$

The linearized boundary condition corresponding to Eq. (1.26a) with Eq. (1.26b) on an interface of a gas and its condensed phase is

$$\begin{aligned} \phi(x_i, \zeta_i, \bar{t}) &= \alpha_c \left[ \omega_w + 2\zeta_j u_{wj} + \left( \zeta_j^2 - \frac{3}{2} \right) \tau_w \right] \\ &+ (1 - \alpha_c) \left\{ (1 - \alpha) \left[ \phi(x_i, \zeta_i - 2(\zeta_j - u_{wj})n_j n_i, \bar{t}) + 4\zeta_j u_{wk} n_j n_k \right] \right. \\ &\left. + \alpha \left[ \sigma_w + 2\zeta_j u_{wj} + \left( \zeta_j^2 - \frac{3}{2} \right) \tau_w \right] \right\}, \quad [(\zeta_j - u_{wj})n_j > 0]. \end{aligned} \quad (1.57)$$

## Chapter 2

# Free Molecular Gas: Highly Rarefied Gas Flows

With increase of the Knudsen number of the system, the effect of intermolecular collisions becomes less important. When their effect can be neglected, the gas (its behavior) is called a free molecular gas (free molecular flow). In this chapter, we discuss the free molecular flow when external force is absent.

### 2.1 General solution of free molecular flow

In a free molecular flow, the Boltzmann equation (1.12) reduces to a simple form without the collision term:

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial X_i} = 0. \quad (2.1)$$

Its general solution is simply obtained as follows:

$$f(X_i, \xi_i, t) = f(X_i - \xi_i(t - t_0), \xi_i, t_0), \quad (2.2)$$

where  $t_0$  is a parameter.

### 2.2 Initial-value problem

Consider a gas in an infinite domain without any boundary (body) in the domain. Let the state (the velocity distribution function) of the gas at time  $t = t_0$  be given as

$$f(X_i, \xi_i, t_0) = g(X_i, \xi_i), \quad (2.3)$$

where  $g(X_i, \xi_i)$  is a given function. Then the behavior of the gas in the later time is given as follows:

$$f(X_i, \xi_i, t) = g(X_i - \xi_i(t - t_0), \xi_i). \quad (2.4)$$

By substitution of the solution (2.4) into Eqs. (1.2)–(1.8), the macroscopic variables, such as density, flow velocity, and temperature, can be obtained.

## 2.3 Boundary-value problem

Consider the boundary-value problem in a steady (time-independent) state. Let the boundary conditions (1.24) and (1.27) be written in a common form as follows:

$$f(X_i, \xi_i) = g(X_i, \xi_i) + \int_{\tilde{\xi}_j n_j < 0} \hat{K}(\xi_i, \tilde{\xi}_i, X_i) f(X_i, \tilde{\xi}_i) d\tilde{\xi}, \quad [\xi_j n_j(X_i) > 0], \quad (2.5)$$

where the  $v_{wi} n_i$  terms in (1.24) and (1.27) have dropped off, because the velocity  $v_{wi}$  of the boundary must satisfy the condition  $v_{wi} n_i = 0$  for the system to be steady. According to the general solution (2.2), the velocity distribution function at an arbitrary point in the gas can be expressed by the boundary data of the velocity distribution function. That is, let  $X_{Bi}(X_i, \xi_i/\xi)$  [ $\xi = (\xi_i^2)^{1/2}$ ] be the point on the boundary (including the infinity) encountered for the first time when we trace back the path of the molecule of velocity  $\xi_i$  (or trace in  $-\xi_i$  direction) from  $X_i$ . Then,

$$f(X_i, \xi_i) = f(X_{Bi}(X_i, \xi_i/\xi), \xi_i), \quad (2.6)$$

where  $\xi_j n_j(X_{Bi}) > 0$  with  $n_i(X_{Bi})$  being the unit normal vector to the boundary at  $X_{Bi}$  and pointed to the gas domain. Thus, the problem is reduced to obtaining the velocity distribution function on the boundary of the molecules with  $\xi_j n_j(X_{Bi}) > 0$  (the reflected molecules).

### 2.3.1 Free molecular gas around a convex body

Consider the case where a convex body lies solely in an infinite expanse of a gas. All the molecules impinging on a point  $X_i$  on the boundary [those with  $\xi_j n_j(X_i) < 0$ ] of the body come from infinity ( $X_{Bi}$  is at infinity). The velocity distribution function for these molecules is determined by the boundary condition at infinity:

$$f(X_i, \xi_i) = f(\infty(X_i, \xi_i/\xi), \xi_i), \quad [\xi_j n_j(X_i) < 0], \quad (2.7)$$

where the arguments of  $\infty$  are shown for discrimination, because the condition at infinity may not be uniform. From Eq. (2.7) the velocity distribution function for the reflected molecules [ $\xi_j n_j(X_i) > 0$ ] at the boundary point  $X_i$  is given with the aid of the boundary condition (2.5) as follows:

$$f(X_i, \xi_i) = g(X_i, \xi_i) + \int_{\tilde{\xi}_j n_j < 0} \hat{K}(\xi_i, \tilde{\xi}_i, X_i) f(\infty(X_i, \tilde{\xi}_i/\xi), \tilde{\xi}_i) d\tilde{\xi}, \quad [\xi_j n_j(X_i) > 0]. \quad (2.8)$$

Thus, knowing the condition at infinity, we can obtain the velocity distribution function of the gas molecules (and therefore the macroscopic variables by integration) at an arbitrary point in the gas with the aid of Eqs. (2.6) and (2.8) [Use  $f(X_i, \xi_i)$  given by Eq. (2.8) as the boundary velocity distribution function on the right hand side of Eq. (2.6)]. Various examples are given in Heineman [1948].

### 2.3.2 General case

Consider a system where a nonconvex body or several convex or nonconvex bodies lie in a gas (bounded or unbounded). Replacing the velocity distribution function  $f(X_i, \xi_i)$  of the impinging molecules under the integral sign in the boundary condition (2.5) by that of the reflected molecules on other parts of the boundary, we obtain an integral equation of the velocity distribution function for the reflected molecules on the boundary:

$$f(X_i, \xi_i) = g(X_i, \xi_i) + \int_{\tilde{\xi}_j n_j < 0} \hat{K}(\xi_i, \tilde{\xi}_i, X_i) f(X_{Bi}(X_i, \tilde{\xi}_i/\tilde{\xi}), \tilde{\xi}_i) d\tilde{\xi},$$

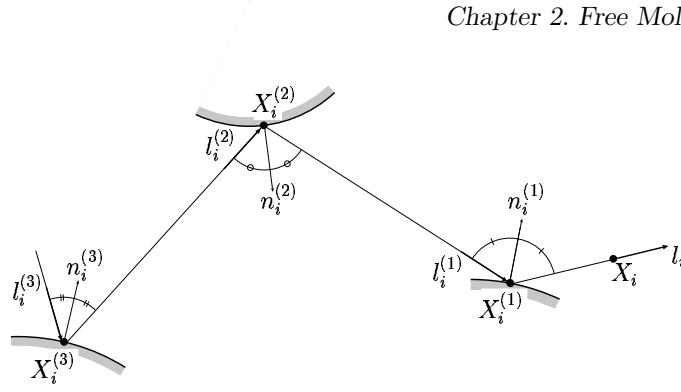
$$[\xi_j n_j(X_i) > 0]. \quad (2.9)$$

It is noted that  $X_i$  in this equation runs only on the boundary.

In the case of the diffuse-reflection boundary condition, we can reduce the integral equation (2.9) for the boundary data of the velocity distribution function for  $\xi_i n_i > 0$  to the integral equation for  $\bar{\rho}_w$ , which is defined by Eq. (1.23b) in the diffuse reflection condition and is independent of  $\xi_i$ . This is done directly from the diffuse-reflection condition as follows. In Eq. (1.23b) (the definition of  $\bar{\rho}_w$ ), the velocity distribution function  $f$  of the impinging molecules under the integral sign is replaced by the right-hand side of Eq. (2.6), which is given by Eq. (1.23a) with  $\alpha = 1$  (the diffuse-reflection condition) and thus is expressed with  $\bar{\rho}_w$ . Thus, the integral equation for  $\bar{\rho}_w$  is obtained.

## 2.4 Statics of a free molecular gas: Effect of the temperature of the boundary

Consider the time-independent behavior of a free molecular gas around a group of bodies at rest. The gas region may be bounded or extend to infinity. The shape and arrangement of the boundary (bodies or outer wall) and the temperature distribution on the boundary are arbitrary. We assume that the reflected gas molecules on the boundary satisfy the Maxwell-type condition [(1.23a) and (1.23b)]. In the case where the domain extends to infinity, the condition there is reserved for a moment. A simple way to solve this problem is developed in Sone [1984a, 1985]. In this section we will introduce the method and its applications without proof or derivation.



**Figure 2.1.** Path of a specularly reflecting molecule and  $X_i^{(m)}$ ,  $l_i^{(m)}$ , and  $n_i^{(m)}$ .

### 2.4.1 Velocity distribution function

The velocity distribution function of the system is given by a series consisting of the data of the temperature and accommodation coefficient of the boundary points that are encountered when we trace back from the point under consideration along the path of a molecule that makes the specular reflection on the boundary. That is, the velocity distribution function  $f(X_i; \xi, l_i)$  [ $= f(X_i, \xi_i; l_i = \xi_i/\xi)$ ] is given in the following series:

$$\begin{aligned} f(X_i; \xi, l_i) &= \alpha^{(1)} G^{(1)} + (1 - \alpha^{(1)}) \alpha^{(2)} G^{(2)} \\ &\quad + (1 - \alpha^{(1)})(1 - \alpha^{(2)}) \alpha^{(3)} G^{(3)} + \dots \\ &= \sum_{m=1}^{\infty} \prod_{h=1}^{m-1} (1 - \alpha^{(h)}) \alpha^{(m)} G^{(m)}, \end{aligned} \quad (2.10)$$

$$\left. \begin{aligned} \alpha^{(m)} &= \alpha(X_i^{(m)}), & G^{(m)} &= C \beta_{(m)}^2 \exp(-\beta_{(m)} \xi^2), \\ \beta_{(m)} &= \beta(X_i^{(m)}) = \frac{1}{2RT_w(X_i^{(m)})}, & C &= \text{const}, \end{aligned} \right\} \quad (2.11)$$

where the series  $X_i^{(m)}$  ( $m = 1, 2, 3, \dots$ ) are the points successively encountered when we trace back the path of a specularly reflecting molecule with velocity  $\xi_i$  from the point  $X_i$ . These are obtained from  $X_i$  and  $l_i$  successively as follows: The  $X_i^{(m)}$  is the first intersection of the boundary and the semi-straight line  $X_i^{(m-1)} - l_i^{(m-1)}s$ , where  $s$  is a positive parameter, and

$$l_i^{(m)} = l_i^{(m-1)} - 2l_j^{(m-1)} n_j^{(m)} n_i^{(m)}, \quad (2.12)$$

where  $X_i^{(0)} = X_i$ ,  $l_i^{(0)} = l_i$ , and  $n_i^{(m)} = n_i(X_j^{(m)})$  (Fig. 2.1). If  $|X_i^{(m)}| = \infty$  at some  $m$ , then the series (2.10) is terminated at this term. Incidentally, from Eq. (2.12) and the relation  $l_i n_i^{(1)} > 0$ , we have

$$|l_i^{(m)}| = 1, \quad l_i^{(m)} n_i^{(m)} < 0, \quad l_i^{(m)} n_i^{(m+1)} > 0. \quad (2.13)$$

Further, we use the following conventions in Eq. (2.10): (i)  $\prod_{h=1}^0 (1 - \alpha^{(h)}) = 1$ , (ii)  $\alpha(\infty) = 1$ , (iii) The infinity may not be uniform. That is,  $\beta(\infty)$  may depend on  $l_i$  and the position at infinity.

The series (2.10) converges uniformly with respect to  $X_i$  and  $l_i$  when  $0 < \varepsilon \leq \alpha \leq 1$  and  $0 < \delta \leq T_w$  ( $\varepsilon$  and  $\delta$  are constants). The error estimate of truncated series is easy, and the series converges rapidly when  $\alpha$  is not too small. In the case of the diffuse reflection ( $\alpha = 1$ ), only the first term of the series remains:

$$f(X_i; \xi, l_i) = G^{(1)}. \quad (2.14)$$

When the domain extends to infinity, it is seen from Eq. (2.10) and the conventions (ii) and (iii) that the following condition should be satisfied at infinity for the series (2.10) to be the solution of the problem: the velocity distribution function for the molecules leaving infinity is given by

$$f = C\beta^2 \exp(-\beta\xi^2), \quad (2.15)$$

where  $\beta$  may depends on  $l_i$  and the position at infinity.

The constant  $C$  in Eq. (2.10) is determined by the density at infinity or the total mass of the system for a finite domain system.

### 2.4.2 Macroscopic variables

Macroscopic variables, such as density, flow velocity, and temperature, are expressed by some moments of the velocity distribution function. Three dimensional integrations there being carried out in a polar coordinate variables, the integration with respect to  $\xi$  can be carried out without specifying the problem explicitly as follows:

$$\begin{aligned} & \int \xi_{i_1} \xi_{i_2} \cdots \xi_{i_h} f \xi^2 d\xi d\Omega \\ &= CE_{h+2} \int l_{i_1} l_{i_2} \cdots l_{i_h} \left( \alpha^{(1)} \beta_{(1)}^{(1-h)/2} + (1 - \alpha^{(1)}) \alpha^{(2)} \beta_{(2)}^{(1-h)/2} \right. \\ & \quad \left. + (1 - \alpha^{(1)})(1 - \alpha^{(2)}) \alpha^{(3)} \beta_{(3)}^{(1-h)/2} + \cdots \right) d\Omega, \end{aligned} \quad (2.16)$$

where

$$E_h = \int_0^\infty \xi^h \exp(-\xi^2) d\xi,$$

and  $d\Omega$  is the solid angle element in the molecular velocity space.

### 2.4.3 Flow velocity

Flow velocity, which corresponds to the case  $h = 1$  in Eq. (2.16), is found to be zero with the aid of the formula  $\alpha^{(1)} + (1 - \alpha^{(1)}) \alpha^{(2)} + (1 - \alpha^{(1)})(1 - \alpha^{(2)}) \alpha^{(3)} +$

$\dots = 1$ , which is easily found by replacing the last factor  $\alpha^{(n)}$  in each term of the series by  $1 - (1 - \alpha^{(n)})$  and summing up the series. Thus, in any system which is enclosed by an outer boundary (or closed system) or in an open system where infinities are in a common equilibrium state at rest, no flow is induced for any distribution of temperature on the bodies. This is not an obvious matter. In a rarefied gas, it is known that various kinds of flows are induced by temperature field, such as thermal creep flow, thermal stress flow (thermal stress slip flow or nonlinear thermal stress flow), flow induced around the edge of a heated plate, thermal transpiration flow (see Chapters 3 and 4 or Sone[2000a]). The above-mentioned result shows that these flows induced owing to gas rarefaction vanish in the limit of high gas rarefaction. Even in the absence of flow, as will be shown in Sections 2.4.5.B and C, an interaction force generally acts between bodies with different temperatures or propulsion force acts on a nonconvex heated body. In the classical fluid dynamics, no steady flow is induced in the situation under consideration  $[(v_i = 0, p = \text{const.})]$  is the obvious solution of the Navier–Stokes set of equations], and no force acts on a body in the gas.

#### 2.4.4 Principle of superposition

Let the boundary (body, surrounding wall, infinity) consists of  $n$  parts (or bodies)  $B_1, B_2, \dots, B_n$ . The effect of temperature of each boundary  $B_m$  can be expressed separately. Let  $f$  be the solution of the original problem; let  $f_m$  ( $m = 1, 2, \dots, n$ ) be the solution on the same geometrical problem where the temperature distribution on the boundary  $B_m$  is kept at the original one, but the temperature distribution on the other boundaries ( $B_1, B_2, \dots, B_n$  except  $B_m$ ) is changed to a uniform temperature  $T_0$ ; and  $f_0$  be the solution of the problem where all the boundaries are set at a uniform temperature  $T_0$ . Then, the solution  $f$  is expressed by  $f_m$  and  $f_0$  under the condition that the accommodation coefficient does not depend on temperature, though it may depend on the position, as follows:

$$f = f_1 + f_2 + \dots + f_n - (n - 1)f_0. \quad (2.17)$$

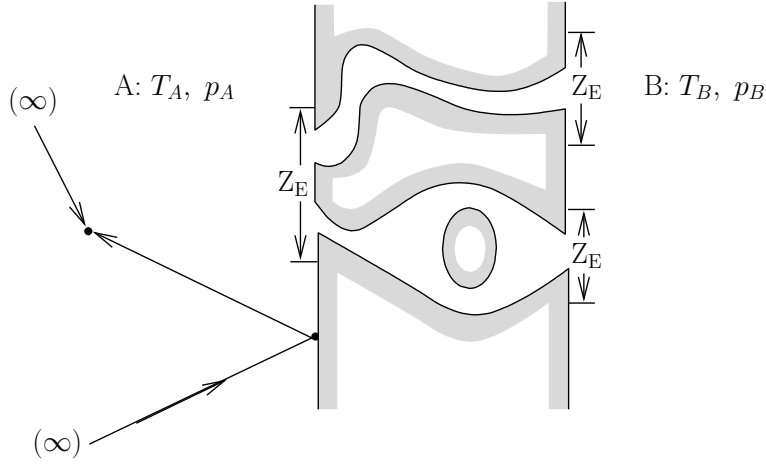
The force  $F_i^{(s)}$  acting on the boundary  $B_s$  is given by

$$F_i^{(s)} = F_i^{(s,1)} + F_i^{(s,2)} + \dots + F_i^{(s,n)}, \quad (2.18)$$

where  $F_i^{(s,m)}$  is the force acting on the boundary  $B_s$  in the situation corresponding to  $f_m$ . The contribution of  $f_0$  term vanishes because  $f_0$  is a uniform Maxwellian.

#### 2.4.5 Application

Various problems are studied on the basis of the solution (2.10). They are found in Sone [1984a, 1985], Sone, Aoki & Ohwada [1986], Aoki, Sone & Ohwada [1986], Sone & Tanaka [1986], and Sone & Aoki [1994]. Here we will give a few of them.



**Figure 2.2.** Two reservoirs joined by pipes. The temperature of the wall of reservoir A is  $T_A$  and that of the reservoir B is  $T_B$  except in a finite region  $Z_E$  in the neighborhood of the entrance of the pipes.

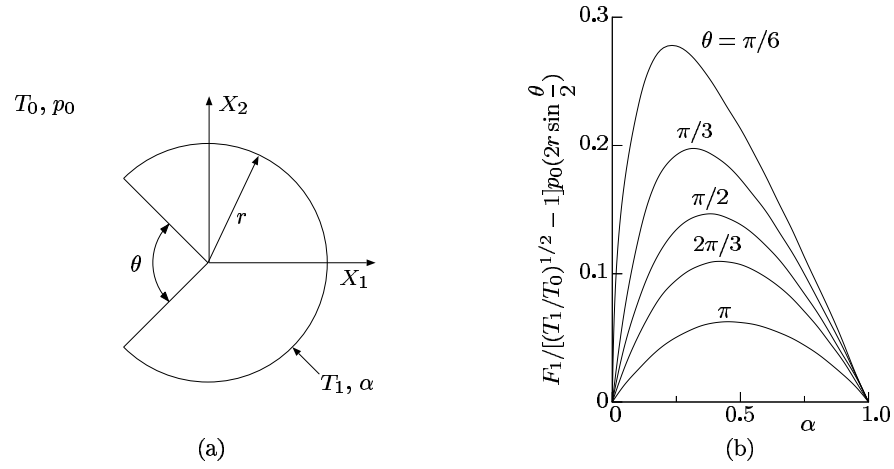
### A Reservoirs joined by various pipes

Equation (2.15) is an additional condition for the series (2.10) to be the solution in the case where the domain extends to infinity, and thus limits the applicability of the solution. In return, from this restriction, we can derive the condition that keeps a stationary state between reservoirs with different temperatures. Two large reservoirs, one, say A, in equilibrium state at temperature  $T_A$  and pressure  $p_A$  and the other, say B, in equilibrium state at temperature  $T_B$  and pressure  $p_B$  are joined by pipes (Fig. 2.2). The temperature of the reservoir wall may differ from  $T_A$  on the side of the reservoir A or from  $T_B$  on the side of B in a finite region in the neighborhood of the entrance of the pipes. When the solution (2.10) is applicable, or the state of the gas at infinity is given by Eq. (2.15), no flow occurs in this system. In the present case, the velocity distribution function at infinity is compatible with Eq. (2.15), if  $\beta = 1/2RT_A$  in A and  $\beta = 1/2RT_B$  in B. Then, the pressure  $p_A$  in A and that  $p_B$  in B are given, respectively, by  $\pi^{3/2}C\sqrt{RT_A/2}$  and  $\pi^{3/2}C\sqrt{RT_B/2}$ . Eliminating  $C$  from these relations, we obtain the condition under which no flow occurs between reservoirs A and B as follows:

$$\frac{p_A}{\sqrt{T_A}} = \frac{p_B}{\sqrt{T_B}}. \quad (2.19)$$

This relation applies irrespective of the condition of the pipes, i.e., the shape and number of the pipes or the distributions of temperature and accommodation coefficient along the pipes. The condition (2.19) is known for a long time for two special cases where the connection between the reservoirs is a slit or a straight pipe with specularly reflecting wall.





**Figure 2.3.** Cylindrical shell and the force acting on it.

### B Propulsion force on a heated body

Consider a uniformly heated body with uniform accommodation coefficient in a uniform gas at rest. If the body shape is convex, it is easily seen that neither force nor moment of force acts on the body. In the case of a nonconvex body, the situation is different and the body is generally subject to a force. As an example, the force acting on a uniformly heated cylindrical shell [Fig. 2.3 (a)] is shown in Fig. 2.3 (b), where  $F_i$  is the force per unit length of the shell,  $T_1$ ,  $\alpha$ ,  $r$ , and  $\theta$  are, respectively, the temperature, accommodation coefficient, radius, and opening angle of the cylindrical shell, and  $T_0$  and  $p_0$  are, respectively, the temperature and pressure of the gas at infinity.

### C Interaction force between heated bodies

If there are two or more heated bodies in a gas, interaction forces act between them. Some simple propositions hold when the gas molecules make the diffuse reflection on the bodies. Consider a system where a group of bodies lie in an infinite expanse of a uniform stationary gas in equilibrium at temperature  $T_0$  and pressure  $p_0$ . The gas molecules make the diffuse reflection on the bodies. Then the following propositions hold:

**[Proposition 2.4.1]** If all the bodies are heated (or cooled) at the same uniform temperature (say,  $T_1$ ), neither force nor moment of force acts on the system of the bodies as a whole.

**[Proposition 2.4.2]** The force and moment of force on a body in the system do not depend on its own temperature.

The proof of the propositions is given in Sone & Aoki [1994].

As examples of interaction force of a specific problem, simple formulas for two parallel cylinders and for two spheres are given here. Two parallel cylinders

1 and 2 (or spheres 1 and 2) lie in a uniform stationary gas at temperature  $T_0$  and pressure  $p_0$ . The cylinder 1 (or sphere 1) has a uniform surface temperature  $T_1$  and radius  $r_1$ , and the other  $T_2$  and  $r_2$ . The distance between the axes (or centers) of the two cylinders (or spheres) is  $D$ . Then, cylinder 1 (per unit length) is subject to a force in the direction from the axis of cylinder 2 to that of cylinder 1 and normal to the axes with the following magnitude  $F$  (including sign):

$$F = (4r_1r_2/\pi D) p_0(\sqrt{T_2/T_0} - 1). \quad (2.20)$$

Sphere 1 is subject to a force in the direction from the center of sphere 2 to that of sphere 1 with the following magnitude  $F$  (including sign):

$$F = (3\pi/4)(r_1r_2/D)^2 p_0(\sqrt{T_2/T_0} - 1). \quad (2.21)$$

The negative value of  $F$  in these formulas means that the direction of force is from cylinder 1 (sphere 1) to cylinder 2 (sphere 2).



## Chapter 3

# Asymptotic Theory of the Boltzmann System for Small Knudsen Numbers: Slightly Rarefied Gas Flows

When the Knudsen number becomes smaller, the contribution of the collision term in the Boltzmann equation becomes larger, and the velocity distribution function will approach a local Maxwellian. Then, the behavior of the gas may be considered to admit a macroscopic description because the distribution function is determined by the five macroscopic variables. In fact, the asymptotic theory of the Boltzmann system for small Knudsen numbers is developed for a time-independent boundary-value problem in an arbitrary domain by Sone [1969,1971,1991a], Darrozes [1969], Sone & Aoki [1987], and Aoki & Sone [1991].<sup>1</sup> According to it, the overall behavior of the gas is described by a fluid-dynamic-type equation with a boundary condition given by a prescribed formula. The corrections are required in a thin layer near the boundary and in a shock layer. Their formulas are established.

In this chapter we consider the time-independent boundary-value problem of the Boltzmann equation in an arbitrary domain except that the shape of the boundary is smooth, and investigate the asymptotic behavior of the solution for small Knudsen numbers. Besides the fundamental formulas mentioned above, i.e., fluid-dynamic-type equations and their associate boundary conditions where the effect of gas rarefaction is taken into account, what is specially to be mentioned here is that there are important classes of problems where the classical gas dynamics is incomplete in describing the behavior of a gas in the

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<sup>1</sup>Asymptotic analysis of the initial-value problem of the Boltzmann equation in an infinite domain for the small mean-free-time limit is developed by Grad [1963].

continuum limit.<sup>2</sup> As in the previous chapter, we here limit our consideration to the case where an external force on the gas is absent.

## 3.1 Linear theory

### 3.1.1 Problem

First, we consider the case where the behavior of the gas deviates only slightly from a uniform equilibrium state at rest, and develop the asymptotic theory on the basis of the linearized Boltzmann equation (1.51). The basis equation is

$$\zeta_i \frac{\partial \phi}{\partial x_i} = \frac{1}{k} \mathcal{L}(\phi). \quad (3.1)$$

The boundary condition [(1.24) or (1.27)] is here expressed in an abstract form:

$$\phi = \phi_w, \quad (\zeta_i n_i > 0), \quad (3.2)$$

on the boundary, where  $\phi_w$  may depend on  $\phi$  ( $\zeta_i n_i < 0$ ) linearly.

We will investigate the asymptotic behavior of  $\phi$  for small  $k$ . The method of analysis is due to Sone [1969, 1971] (see also Sone [1984b, 1991] and Sone & Aoki [1994]).

### 3.1.2 Grad–Hilbert expansion

Putting aside the boundary condition, we look for a moderately varying solution of Eq. (3.1), whose length scale of variation is the order of the characteristic length  $L$  of the system [ $\partial \phi / \partial x_i = O(\phi)$ ], in a power series of  $k$ :

$$\phi_G = \phi_{G0} + \phi_{G1}k + \phi_{G2}k^2 + \cdots, \quad (3.3)$$

where the subscript  $G$  is attached for discrimination. Corresponding to this expansion, the macroscopic variables  $\omega, u_i, \tau$ , etc. [Eq. (1.52)] are also expanded in  $k$ :

$$h_G = h_{G0} + h_{G1}k + h_{G2}k^2 + \cdots, \quad (3.4)$$

where  $h = \omega, u_i, \dots$ . The  $h_{Gm}$  is related to  $\phi_{Gm}$  by Eq. (1.52) with  $\phi = \phi_{Gm}$ . Substituting the series (3.3) into the linearized Boltzmann equation (3.1) and

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<sup>2</sup>The “continuum limit” means the limit where the mean free path tends to zero. There is a confusion in some textbooks of classical gas dynamics, where the motion of a gas in the continuum limit is discussed. They confuse the gas under their discussion with a wider class of a gas treated in kinetic theory, in which the number of molecules is so large that the macroscopic variables, such as the density, flow velocity, and temperature, can be defined as continuous functions of position and time. Thus they classify rarefied gas dynamics (or the system described by the Boltzmann equation) as a subject where the behavior of ensemble of much less number of molecules is discussed. This wrong explanation at introductory part of textbooks spreads long lasting serious misunderstanding among fluid-dynamicists.

arranging the terms by the order of  $k$ , we obtain a series of integral equations for  $\phi_{Gm}$ :

$$\mathcal{L}(\phi_{G0}) = 0, \quad (3.5)$$

$$\mathcal{L}(\phi_{Gm}) = \zeta_i \frac{\partial \phi_{Gm-1}}{\partial x_i}, \quad (m = 1, 2, 3 \dots). \quad (3.6)$$

The homogeneous equation (3.5) has five independent solutions  $1, \zeta_i$ , and  $\zeta_j^2$  (Grad [1958], Cercignani [1987], Sone & Aoki [1994]). In view of the relation (1.52) with  $\phi = \phi_{G0}$ , the solution  $\phi_{G0}$  is expressed as

$$\phi_{G0} = \omega_{G0} + 2\zeta_i u_{iG0} + \left( \zeta_j^2 - \frac{3}{2} \right) \tau_{G0}. \quad (3.7)$$

For the inhomogeneous equation (3.6) to have a solution, its inhomogeneous term should satisfy the solvability condition:

$$\int g \zeta_i \frac{\partial \phi_{Gm-1}}{\partial x_i} E d\boldsymbol{\zeta} = 0, \quad d\boldsymbol{\zeta} = d\zeta_1 d\zeta_2 d\zeta_3, \quad (3.8)$$

where

$$g = 1, \zeta_j, \text{ or } \zeta_j^2, \quad (3.9)$$

because  $\int g \mathcal{L}(\phi) E d\boldsymbol{\zeta} = 0$  (Grad [1958], Cercignani [1987], Sone & Aoki [1994]). Then, the solution  $\phi_{Gm}$  is given in the form:

$$\phi_{Gm} = \omega_{Gm} + 2\zeta_i u_{iGm} + \left( \zeta_j^2 - \frac{3}{2} \right) \tau_{Gm} + \hat{\phi}_{Gm}, \quad (3.10)$$

where  $\hat{\phi}_{Gm}$  is the particular solution of Eq. (3.6) orthogonal to  $g$ :

$$\int \hat{\phi}_{Gm} g E d\boldsymbol{\zeta} = 0. \quad (3.11)$$

From the integral equation (3.6) up to the  $m$ -th order, together with the solutions (3.7) and (3.10) and the orthogonal relation (3.11), it is seen that  $\hat{\phi}_{Gm}$  is expressed in a linear combination of the  $(m-s)$ -th partial derivatives of  $\omega_{Gs}$ ,  $u_{iGs}$ , and  $\tau_{Gs}$  with respect to  $x_j$  ( $j = 1, 2$ , or  $3$ ;  $s = 0, 1, \dots, m-1$ ), whose coefficients are functions of  $\zeta_i$ . Further, from isotropic property of the collision operator  $\mathcal{L}$  (see Appendix I in Sone & Aoki [1994]), these coefficient functions are polynomials of  $\zeta_i$  ( $i = 1, 2$ , and  $3$ ), whose coefficients are functions of  $\zeta$ . Substituting this  $\phi_{Gm}$  (without the explicit form of the latter coefficient functions) into the solvability condition (3.8), we obtain the following Stokes set of partial differential equations for  $\omega_{Gm}$ ,  $u_{iGm}$ , and  $\tau_{Gm}$ :<sup>3</sup>

$$\frac{\partial P_{G0}}{\partial x_i} = 0, \quad (3.12)$$

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<sup>3</sup>To obtain the equations for  $u_{iGn}$ ,  $\tau_{Gn}$ , and  $P_{Gn+1}$ , the solvability conditions (3.8) up to  $m = n + 2$  are required.

$$\frac{\partial u_{iGm}}{\partial x_i} = 0, \quad (3.13a)$$

$$\frac{\partial P_{Gm+1}}{\partial x_i} = \gamma_1 \frac{\partial^2 u_{iGm}}{\partial x_j^2}, \quad (3.13b)$$

$$\frac{\partial^2 \tau_{Gm}}{\partial x_j^2} = 0, \quad (3.13c)$$

$$(m = 0, 1, 2, \dots),$$

where

$$P_{Gm} = \omega_{Gm} + \tau_{Gm}, \quad (3.14)$$

and  $\gamma_1$  is a constant related to the collision integral and to be given in Eq. (3.27).

When the solvability condition is satisfied, the solution  $\phi_{Gm}$  is given in the form:

$$\phi_{G0} = \phi_{eG0}, \quad (3.15)$$

$$\phi_{G1} = \phi_{eG1} - \zeta_i \zeta_j B(\zeta) \frac{\partial u_{iG0}}{\partial x_j} - \zeta_i A(\zeta) \frac{\partial \tau_{G0}}{\partial x_i}, \quad (3.16)$$

$$\begin{aligned} \phi_{G2} = & \phi_{eG2} - \zeta_i \zeta_j B(\zeta) \frac{\partial u_{iG1}}{\partial x_j} - \zeta_i A(\zeta) \frac{\partial \tau_{G1}}{\partial x_i} + \frac{1}{\gamma_1} \zeta_i D_1(\zeta) \frac{\partial P_{G1}}{\partial x_i} \\ & + \zeta_i \zeta_j \zeta_k D_2(\zeta) \frac{\partial^2 u_{iG0}}{\partial x_j \partial x_k} - \zeta_i \zeta_j F(\zeta) \frac{\partial^2 \tau_{G0}}{\partial x_i \partial x_j}, \end{aligned} \quad (3.17)$$

where

$$\phi_{eGm} = \omega_{Gm} + 2\zeta_i u_{iGm} + \left( \zeta^2 - \frac{3}{2} \right) \tau_{Gm}, \quad (3.18)$$

and the functions  $A(\zeta)$ ,  $B(\zeta)$ ,  $D_1(\zeta)$ ,  $D_2(\zeta)$ , and  $F(\zeta)$  are the solutions of the following integral equations:

$$\left. \begin{aligned} \mathcal{L}[\zeta_i A(\zeta)] &= -\zeta_i \left( \zeta^2 - \frac{5}{2} \right), \\ \text{subsidiary condition : } &\int_0^\infty \zeta^4 A(\zeta) \exp(-\zeta^2) d\zeta = 0, \end{aligned} \right\} \quad (3.19)$$

$$\mathcal{L} \left[ \left( \zeta_i \zeta_j - \frac{1}{3} \zeta^2 \delta_{ij} \right) B(\zeta) \right] = -2 \left( \zeta_i \zeta_j - \frac{1}{3} \zeta^2 \delta_{ij} \right), \quad (3.20)$$

$$\mathcal{L} \left[ \left( \zeta_i \zeta_j - \frac{1}{3} \zeta^2 \delta_{ij} \right) F(\zeta) \right] = \left( \zeta_i \zeta_j - \frac{1}{3} \zeta^2 \delta_{ij} \right) A(\zeta), \quad (3.21)$$

$$\left. \begin{aligned} \mathcal{L}[(\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) D_1(\zeta) + \zeta_i \zeta_j \zeta_k D_2(\zeta)] \\ = \gamma_1 (\zeta_i \delta_{jk} + \zeta_j \delta_{ki} + \zeta_k \delta_{ij}) - \zeta_i \zeta_j \zeta_k B(\zeta), \\ \text{subsidiary condition : } &\int_0^\infty [5\zeta^4 D_1(\zeta) + \zeta^6 D_2(\zeta)] \exp(-\zeta^2) d\zeta = 0. \end{aligned} \right\} \quad (3.22)$$

The numerical data of  $A(\zeta)$ ,  $B(\zeta)$ ,  $D_1(\zeta)$ ,  $D_2(\zeta)$ , and  $F(\zeta)$  for a hard-sphere molecular gas are given in Ohwada & Sone [1992] (The function  $\nu(\zeta)$  in Eqs.

(A1)–(A5) on page 409 in Ohwada & Sone [1992] is a misprint for  $2\sqrt{2}\nu(\zeta)$ . See also Pekeris & Alterman [1957]).<sup>4</sup> For the BKW model,

$$A(\zeta) = \zeta^2 - \frac{5}{2}, \quad B(\zeta) = 2, \quad D_1(\zeta) = -1, \quad D_2(\zeta) = 2, \quad F(\zeta) = -\zeta^2 + \frac{5}{2}. \quad (3.23)$$

### 3.1.3 Stress tensor and heat-flow vector of the Grad–Hilbert solution

From the solution  $\phi_{Gm}$  in Eqs. (3.15)–(3.17), with the aid of the self-adjoint property  $\int [\psi \mathcal{L}(\phi) - \phi \mathcal{L}(\psi)] E \, d\mathbf{\zeta} = 0$ , where  $\psi = \zeta_i A(\zeta)$  or  $(\zeta_i \zeta_j - \frac{1}{3} \zeta^2 \delta_{ij}) B(\zeta)$  and Eqs. (3.19) and (3.20) are applied, we obtain the stress tensor and heat-flow vector of the Grad–Hilbert solution as follows:

$$\left. \begin{aligned} P_{ijG0} &= P_{G0} \delta_{ij}, & P_{ijG1} &= P_{G1} \delta_{ij} + \gamma_1 S_{ijG0}, \\ P_{ijG2} &= P_{G2} \delta_{ij} + \gamma_1 S_{ijG1} + \gamma_3 \frac{\partial^2 \tau_{G0}}{\partial x_i \partial x_j}, \\ P_{ijG3} &= P_{G3} \delta_{ij} + \gamma_1 S_{ijG2} + \gamma_3 \frac{\partial^2 \tau_{G1}}{\partial x_i \partial x_j} - \frac{2\gamma_6}{\gamma_1} \frac{\partial^2 P_{G1}}{\partial x_i \partial x_j}, \end{aligned} \right\} \quad (3.24)$$

$$\left. \begin{aligned} Q_{iG0} &= 0, & Q_{iG1} &= \frac{5}{4} \gamma_2 G_{iG0}, \\ Q_{iG2} &= \frac{5}{4} \gamma_2 G_{iG1} + \frac{\gamma_3}{2\gamma_1} \frac{\partial P_{G1}}{\partial x_i}, & Q_{iG3} &= \frac{5}{4} \gamma_2 G_{iG2} + \frac{\gamma_3}{2\gamma_1} \frac{\partial P_{G2}}{\partial x_i}. \end{aligned} \right\} \quad (3.25)$$

Here

$$S_{ijGm} = - \left( \frac{\partial u_{iGm}}{\partial x_j} + \frac{\partial u_{jGm}}{\partial x_i} \right), \quad G_{iGm} = - \frac{\partial \tau_{Gm}}{\partial x_i}, \quad (3.26)$$

and  $\gamma_n$  is expressed by the solutions  $A(\zeta)$ ,  $B(\zeta)$ , etc. of the integral equations (3.19)–(3.22) as<sup>5</sup>

$$\left. \begin{aligned} \gamma_1 &= I_6(B), & \gamma_2 &= 2I_6(A), \\ \gamma_3 &= I_6(AB) = 5I_6(D_1) + I_8(D_2) = -2I_6(F), \\ \gamma_6 &= \frac{1}{2} I_6(BD_1) + \frac{3}{14} I_8(BD_2), \end{aligned} \right\} \quad (3.27)$$

where  $I_n(Z)$ , with  $Z = A, B, \dots$ , is defined by the integral

$$I_n(Z) = \frac{8}{15\sqrt{\pi}} \int_0^\infty \zeta^n Z(\zeta) \exp(-\zeta^2) d\zeta. \quad (3.28)$$

For a hard-sphere molecular gas, they are obtained as (Ohwada & Sone [1992])

$$\left. \begin{aligned} \gamma_1 &= 1.270042427, & \gamma_2 &= 1.922284066, \\ \gamma_3 &= 1.947906335, & \gamma_6 &= 1.419423836. \end{aligned} \right\} \quad (3.29)$$

<sup>4</sup>The functions  $A(\zeta)$ ,  $B(\zeta)$ , etc. depend on the parameter  $U_0/k_B T_0$  (Section 1.9) except for a hard-sphere molecular gas and the BKW model.

<sup>5</sup>The  $\gamma_n$  as well as the functions  $A(\zeta)$ ,  $B(\zeta)$ , etc. depends on the parameter  $U_0/k_B T_0$  (Section 1.9) except for a hard-sphere molecular gas and the BKW model.



For the BKW model,

$$\gamma_n = 1. \quad (3.30)$$

In the formula (3.24), the term proportional to  $S_{ijGm}$  corresponds to viscous stress in the classical fluid dynamics, the higher order term proportional to  $\partial^2 \tau_{Gm} / \partial x_i \partial x_j$  is called thermal stress, and the term proportional to  $\partial^2 P_{G1} / \partial x_i \partial x_j$  may be called pressure stress. At the second and higher order, the heat-flow vector  $Q_{iGm}$  depends on pressure gradient as well as temperature gradient. A numerical example of the heat flow proportional to pressure gradient is given in the Poiseuille flow analysis by Ohwada, Sone & Aoki [1989b].

### 3.1.4 Analysis of Knudsen layer

The Grad–Hilbert expansion obtained in Section 3.1.2 does not have enough freedom to be fitted to the kinetic boundary condition (3.2), because each term of the expansion is a special form of  $\zeta_i$ , i.e., a polynomial of  $\zeta_i$  with its coefficients of functions of  $\zeta$ . Therefore, the behavior of a steady flow of a slightly rarefied gas past bodies (i.e., the asymptotic solution for small Knudsen numbers of a time-independent boundary-value problem of the Boltzmann equation) cannot be described only by the Grad–Hilbert expansion.

The problem is resolved by introducing the Knudsen-layer correction near the boundary. That is, the asymptotic solution is obtained in the form

$$\phi = \phi_G + \phi_K, \quad (3.31)$$

where  $\phi_K$ , called Knudsen-layer correction, makes an appreciable change in the direction normal to the boundary over the distance of the order of the mean free path [ $kn_i \partial \phi_K / \partial x_i = O(\phi_K)$ ] and is appreciable only in a thin layer (Knudsen layer), with thickness of the order of the mean free path, adjacent to the boundary. In the linearized problem, the equation for  $\phi_K$  is the same as Eq. (3.1). That is,

$$\zeta_i \frac{\partial \phi_K}{\partial x_i} = \frac{1}{k} \mathcal{L}(\phi_K), \quad (3.32)$$

which is rewritten with the Knudsen-layer variables  $(\eta, s_1, s_2)$  as

$$\zeta_i n_i \frac{\partial \phi_K}{\partial \eta} = \mathcal{L}(\phi_K) - k \zeta_i \left( \frac{\partial s_1}{\partial x_i} \frac{\partial \phi_K}{\partial s_1} + \frac{\partial s_2}{\partial x_i} \frac{\partial \phi_K}{\partial s_2} \right), \quad (3.33)$$

$$x_i = k \eta n_i(s_1, s_2) + x_{wi}(s_1, s_2), \quad (3.34)$$

where  $x_{wi}$  is the boundary surface,  $\eta$  is a stretched coordinate normal to the boundary,  $s_1$  and  $s_2$  are (unstretched) coordinates within a parallel surface  $\eta = \text{const}$ , and the normal vector  $n_i$  is a function of  $s_1$  and  $s_2$ . The Knudsen-layer correction  $\phi_K$  is also expanded in a power series of  $k$ :

$$\phi_K = \phi_{K0} + \phi_{K1}k + \cdots. \quad (3.35)$$

Each term  $\phi_{Km}$  of the expansion is governed by a spatially one-dimensional (homogeneous or inhomogeneous) linearized Boltzmann equation:

$$\zeta_i n_i \frac{\partial \phi_{K0}}{\partial \eta} = \mathcal{L}(\phi_{K0}), \quad (3.36)$$

$$\zeta_i n_i \frac{\partial \phi_{K1}}{\partial \eta} = \mathcal{L}(\phi_{K1}) - \zeta_i \left[ \left( \frac{\partial s_1}{\partial x_i} \right)_0 \frac{\partial \phi_{K0}}{\partial s_1} + \left( \frac{\partial s_2}{\partial x_i} \right)_0 \frac{\partial \phi_{K0}}{\partial s_2} \right], \quad (3.37)$$

where the parentheses  $( )_0$  with the subscript 0 indicate that the quantity in the parentheses  $( )$  is evaluated at  $\eta = 0$ .

The boundary condition for  $\phi_{Km}$  at  $\eta = 0$  is

$$\phi_{Km} = \phi_{wm} - \phi_{Gm}, \quad (\zeta_i n_i > 0), \quad (3.38)$$

where  $\phi_{wm}$  is defined by<sup>6</sup>

$$\phi_w = \phi_{w0} + \phi_{w1}k + \cdots. \quad (3.39)$$

Because  $\phi_K$  is assumed to be the correction to  $\phi_G$ , the condition at infinity is

$$\phi_{Km} \rightarrow 0, \quad \text{as } \eta \rightarrow \infty, \quad (3.40)$$

where the decay is assumed to be faster than any inverse power of  $\eta$ . This is verified in the existence and uniqueness theorem explained in the next paragraph.

The boundary condition (3.38) contains undetermined boundary values of the macroscopic variables  $u_{iGm}$ ,  $\tau_{Gm}$ , and  $\omega_{Gm}$ , as well as those of the  $(m-s)$ -th partial derivatives of  $u_{iGs}$ ,  $\tau_{Gs}$ , and  $\omega_{Gs}$  ( $s < m$ ) at the previous stages of approximation, through  $\phi_{Gm}$ .<sup>7</sup> The existence and uniqueness theorem (Bardos, Caflisch & Nicolaenko [1986], Cercignani [1986], Coron, Golse & Sulem [1988], Golse & Poupaud [1989]) of the half-space boundary-value problem of the linearized Boltzmann equation shows that the solution  $\phi_{Km}$  exists when and only when the macroscopic variables satisfy some relations. These relations give the boundary condition for the Stokes set of partial differential equations (3.12)–(3.13c). The process of derivation of the boundary condition is explained for the case of the complete-condensation boundary condition in the next paragraph.

The existence and uniqueness theorem for the half-space problem of the linearized Boltzmann equation [Eq. (3.36) or (3.37) and Eqs. (3.38) and (3.40)] by Bardos, Caflisch & Nicolaenko [1986] is as follows. Let the boundary condition for the linearized Boltzmann equation (3.36) or (3.37) at  $\eta = 0$  be given by

$$\phi_{Km} = c_0 + c_i(\zeta_i - \zeta_j n_j n_i) + c_4 \zeta_j^2 + f(\zeta_i), \quad (\zeta_i n_i > 0), \quad (3.41)$$

<sup>6</sup>The boundary value  $\phi_w$  depends on  $k$  because  $\phi_w$  generally depends on  $\phi$  ( $\zeta_i n_i < 0$ ). In some cases, the boundary data  $u_{wi}$ ,  $\tau_w$ , and  $\omega_w$  (thus,  $P_w$ ) are unknown beforehand and may depend on the Knudsen number, for example, the surface temperature of a particle set freely in a gas. To include the case, the data  $u_{wi}$ ,  $\tau_w$ ,  $\omega_w$ , and  $P_w$  also are expanded in power series of  $k$ .

<sup>7</sup>For simplicity, consider the case of the diffuse reflection or the complete condensation condition. In the case of specular reflection condition, the  $u_{iGm} - u_{jGm} n_j n_i$ ,  $\tau_{Gm}$ , and  $\omega_{Gm}$  terms are canceled out in Eq. (3.38). Thus, the discussion in the next paragraph has to be modified, and formulas different from those in Section 3.1.5 are derived (Sone & Aoki [1977a]).

where  $c_0, c_i$ , and  $c_4$  are undetermined constants, which are practically four because of no contribution of  $c_i n_i$  to Eq. (3.41), and  $f(\zeta_i)$  is a given function. Then the solution of this boundary-value problem, where exponential decay of the inhomogeneous term as  $\eta \rightarrow \infty$  is assumed, exists when and only when the four undetermined constants  $c_0, c_i - c_j n_j n_i$ , and  $c_4$  take special values.<sup>8</sup> The solution is unique including  $c_0, c_i - c_j n_j n_i$ , and  $c_4$ . It is also shown that the speed of decay of the solution as  $\eta \rightarrow \infty$  is exponential.<sup>9</sup> In the boundary condition (3.38),  $\omega_{wm} - \omega_{Gm}$ ,  $u_{wim} - (u_{iGm} - u_{jGm} n_j n_i)$ ,  $\tau_{wm} - \tau_{Gm}$ , and  $2\zeta_i u_{jGm} n_i n_j + \hat{\phi}_{Gm}$  correspond, respectively, to  $c_0, c_i - c_j n_j n_i, c_4$ , and  $f(\zeta_i)$ . Thus, the solution  $\phi_{Km}$  and the boundary data  $\omega_{wm} - \omega_{Gm}$ ,  $u_{wim} - (u_{iGm} - u_{jGm} n_j n_i)$ , and  $\tau_{wm} - \tau_{Gm}$  are determined by  $2\zeta_i u_{jGm} n_i n_j + \hat{\phi}_{Gm}$  and  $\phi_{Kr}$  ( $r < m$ ) in the inhomogeneous term of the equation for  $\phi_{Gm}$ , e.g., Eq. (3.37) for  $m = 1$ . In view of the form of  $\hat{\phi}_{Gm}$  [see Eqs. (3.10), (3.15)–(3.17)], the way that  $\phi_{Km}$  is determined from  $\phi_{Kr}$  ( $r < m$ ), and the linear property of the problem, the undetermined constants  $\omega_{wm} - \omega_{Gm}$ ,  $u_{wim} - (u_{iGm} - u_{jGm} n_j n_i)$ , and  $\tau_{wm} - \tau_{Gm}$  and the solution  $\phi_{Km}$  are expressed by the linear combinations of the boundary values of  $u_{jGm} n_i n_j$  and the  $(m - r)$ -th or the lower order partial derivatives of  $\omega_{Gr}$ ,  $u_{iGr}$ , and  $\tau_{Gr}$  ( $r < m$ ), where the coefficients in the expression for  $\phi_{Km}$  are functions of  $\eta$ . The expressions for  $\omega_{wm} - \omega_{Gm}$ ,  $u_{wim} - (u_{iGm} - u_{jGm} n_j n_i)$ , and  $\tau_{wm} - \tau_{Gm}$  correspond to the slip boundary conditions.<sup>10</sup> In the case of the diffuse reflection, further discussion is required, because  $\sigma_{wm}$  term is related to  $\phi_{Gm} + \phi_{Km}$  with  $\zeta_i n_i < 0$  (see Sone & Aoki [1994]). For the Maxwell type boundary condition, we have to use a more general work by Coron, Golse & Sulem [1988].

### 3.1.5 Slip boundary condition and Knudsen-layer correction

Here we summarize the boundary condition for the Stokes set of equations on a simple boundary or on an interface of a gas and its condensed phase and the Knudsen-layer correction to the macroscopic variables of the Grad–Hilbert solution, which are determined by the Stokes set of equations and its associate boundary condition. The formulas apply to a locally isotropic boundary,<sup>11</sup> where the reflection law is expressed by Eq. (1.24) with a finite diffuse-reflection part or by Eq. (1.27) with a finite complete-condensation part.<sup>12</sup> The boundary conditions are called a slip or jump condition.

<sup>8</sup>This is first conjectured by Grad [1969].

<sup>9</sup>This guarantees the exponential decay of the inhomogeneous term in the next order Knudsen-layer equation.

<sup>10</sup>Hereafter,  $P_{wm} - P_{Gm}$ , where  $P_{wm} = \omega_{wm} + \tau_{wm}$  in the linear theory, will be used exclusively instead of  $\omega_{wm} - \omega_{Gm}$ .

<sup>11</sup>The “locally isotropic” means that the scattering kernel  $K_B(\xi_i, \tilde{\xi}_i, X_i)$  or  $K_I(\xi_i, \tilde{\xi}_i, X_i)$  and the inhomogeneous function  $g_B(X_i, \xi_i)$  in the kinetic boundary condition (1.24) or (1.27) are invariant for the following two kinds of transformations of  $\xi_i - v_{wi}$  and  $\tilde{\xi}_i - v_{wi}$  (simply, those of  $\xi_i$  and  $\tilde{\xi}_i$  for the present linear theory): a rotation around the normal  $n_i$  of the boundary and the reflection with respect to a plane containing the normal  $n_i$ .

<sup>12</sup>This is the case where the  $u_{iGm} - u_{jGm} n_j n_i$ ,  $\tau_{Gm}$ , and  $\omega_{Gm}$  terms are not canceled out and remain with finite magnitudes in Eq.(3.38) (see Footnote 7).

### A On a solid simple boundary

On a solid simple boundary, the slip boundary condition and Knudsen-layer correction are given as follows (Sone [1969,1971,1991], Sone & Takata [1992]):<sup>13</sup>

$$u_{iG0} - u_{wi0} = 0, \quad (3.42a)$$

$$\tau_{G0} - \tau_{w0} = 0, \quad (3.42b)$$

$$u_{iK0} = \omega_{K0} = \tau_{K0} = 0, \quad (3.42c)$$

$$\begin{bmatrix} (u_{iG1} - u_{wi1})t_i \\ u_{iK1}t_i \end{bmatrix} = S_{ijG0}n_it_j \begin{bmatrix} k_0 \\ Y_0(\eta) \end{bmatrix} + G_{iG0}t_i \begin{bmatrix} K_1 \\ \frac{1}{2}Y_1(\eta) \end{bmatrix}, \quad (3.43a)$$

$$\begin{bmatrix} u_{iG1}n_i \\ u_{iK1}n_i \end{bmatrix} = 0, \quad (3.43b)$$

$$\begin{bmatrix} \tau_{G1} - \tau_{w1} \\ \omega_{K1} \\ \tau_{K1} \end{bmatrix} = -G_{iG0}n_i \begin{bmatrix} d_1 \\ \Omega_1(\eta) \\ \Theta_1(\eta) \end{bmatrix}, \quad (3.43c)$$

$$\begin{aligned} \begin{bmatrix} (u_{iG2} - u_{wi2})t_i \\ u_{iK2}t_i \end{bmatrix} &= S_{ijG1}n_it_j \begin{bmatrix} k_0 \\ Y_0(\eta) \end{bmatrix} + \frac{\partial S_{ijG0}}{\partial x_r}n_jn_rt_i \begin{bmatrix} a_1 \\ Y_{a1}(\eta) \end{bmatrix} \\ &\quad + \bar{\kappa}S_{ijG0}n_it_j \begin{bmatrix} a_2 \\ Y_{a2}(\eta) \end{bmatrix} + \kappa_{ij}S_{jrg0}n_rt_i \begin{bmatrix} a_3 \\ Y_{a3}(\eta) \end{bmatrix} \\ &\quad + \frac{\partial G_{iG0}}{\partial x_j}n_jt_i \begin{bmatrix} a_4 \\ Y_{a4}(\eta) \end{bmatrix} + \bar{\kappa}G_{iG0}t_i \begin{bmatrix} a_5 \\ Y_{a5}(\eta) \end{bmatrix} \\ &\quad + \kappa_{ij}G_{jG0}t_i \begin{bmatrix} a_6 \\ Y_{a6}(\eta) \end{bmatrix} - \frac{\partial \tau_{w1}}{\partial x_i}t_i \begin{bmatrix} K_1 \\ \frac{1}{2}Y_1(\eta) \end{bmatrix}, \end{aligned} \quad (3.44a)$$

$$\begin{aligned} \begin{bmatrix} u_{iG2}n_i \\ u_{iK2}n_i \end{bmatrix} &= \frac{\partial S_{ijG0}}{\partial x_r}n_in_jn_r \begin{bmatrix} b_1 \\ \frac{1}{2}\int_{-\infty}^{\eta} Y_0(\eta_0)d\eta_0 \end{bmatrix} \\ &\quad + \left( \frac{\partial G_{iG0}}{\partial x_j}n_in_j + 2\bar{\kappa}G_{iG0}n_i \right) \begin{bmatrix} b_2 \\ \frac{1}{2}\int_{-\infty}^{\eta} Y_1(\eta_0)d\eta_0 \end{bmatrix}, \end{aligned} \quad (3.44b)$$

<sup>13a)</sup> There is a misprint in the third equation of the slip boundary condition (24) in Sone [1991], where the term  $-d_3n_in_j\partial G_{iG0}/\partial x_j$  is missing. b) For the specular reflection or the Maxwell-type condition with very small accommodation coefficient, the formulas take different forms (Sone & Aoki [1977a], Aoki, Inamuro & Onishi [1979]) (see Footnote 7). c) In some literature, a temperature jump term proportional to  $S_{ijG0}n_in_j$  is retained in  $\tau_{G1} - \tau_{w1}$  of Eq.(3.43c). However,  $S_{ijG0}n_in_j = 0$  on a solid simple boundary with the aid of Eqs.(3.13a) and (3.42a).

$$\begin{bmatrix} \tau_{G2} - \tau_{w2} \\ \omega_{K2} \\ \tau_{K2} \end{bmatrix} = -G_{iG1}n_i \begin{bmatrix} d_1 \\ \Omega_1(\eta) \\ \Theta_1(\eta) \end{bmatrix} - \frac{\partial S_{ijG0}}{\partial x_r} n_i n_j n_r \begin{bmatrix} d_4 \\ \Omega_{d4}(\eta) \\ \Theta_{d4}(\eta) \end{bmatrix} \\ - \frac{\partial G_{iG0}}{\partial x_j} n_i n_j \begin{bmatrix} d_3 \\ \Omega_{d3}(\eta) \\ \Theta_{d3}(\eta) \end{bmatrix} - \bar{\kappa} G_{iG0} n_i \begin{bmatrix} d_5 \\ \Omega_{d5}(\eta) \\ \Theta_{d5}(\eta) \end{bmatrix}, \quad (3.44c)$$

$$\left. \begin{aligned} \bar{\kappa} &= \frac{1}{2}(\kappa_1 + \kappa_2), & \kappa_{ij} &= \kappa_1 \ell_i \ell_j + \kappa_2 m_i m_j, \\ u_{wi} &= u_{wi0} + u_{wi1}k + \cdots, & \tau_w &= \tau_{w0} + \tau_{w1}k + \cdots, \end{aligned} \right\} \quad (3.45)$$

where  $(2RT_0)^{1/2}u_{wi}$  ( $u_{wi}n_i = 0$ )<sup>14</sup> and  $T_0(1 + \tau_w)$  are the velocity and temperature of the boundary, respectively;  $k_0, K_1, a_1, \dots, a_6, b_1$  [=  $\frac{1}{2} \int_0^\infty Y_0(\eta_0) d\eta_0$ ],  $b_2$  [=  $\frac{1}{2} \int_0^\infty Y_1(\eta_0) d\eta_0$ ],  $d_1, d_3, d_4$ , and  $d_5$  are constants called slip coefficients, determined by the molecular model (e.g., hard sphere, BKW) and the reflection law on the boundary (e.g., diffuse reflection);  $Y_0(\eta)$ ,  $Y_1(\eta)$ ,  $\Omega_1(\eta)$ ,  $\Theta_1(\eta)$ , etc. are functions of  $\eta$ , called Knudsen-layer functions, whose functional form is determined by the molecular model and the reflection law;<sup>15</sup> the  $\kappa_1/L$  and  $\kappa_2/L$  are the principal curvatures of the boundary, where  $\kappa_1$  or  $\kappa_2$  is taken negative when the corresponding center of curvature lies on the side of the gas; the  $\ell_i$  and  $m_i$  are the direction cosines of the principal directions corresponding to  $\kappa_1$  and  $\kappa_2$  respectively;<sup>16</sup> the quantities with the subscript  $G$  are evaluated on the boundary surface.

For a hard-sphere molecular gas under the diffuse reflection, the 1st-order slip coefficients and Knudsen-layer functions and some of them of the 2nd-order are obtained in Ohwada, Sone & Aoki [1989a], Sone, Ohwada & Aoki [1989a], and Ohwada & Sone [1992].<sup>17</sup>

$$\left. \begin{aligned} k_0 &= -1.2540, & K_1 &= -0.6463, & d_1 &= 2.4001, \\ a_4 &= 0.0330, & b_1 &= 0.1068, & b_2 &= 0.4776, \end{aligned} \right\} \quad (3.46)$$

and

$$\left. \begin{aligned} \left( k_0, \eta, Y_0(\eta), \frac{1}{2}Y_1(\eta) \right) &= \left( -\beta_A, x_1, -S(x_1), -C(x_1) \right) \\ &\quad \text{in Ohwada, Sone \& Aoki [1989a],} \\ \left( d_1, \eta, \Omega_1(\eta), \Theta_1(\eta) \right) &= \left( \beta, x_1, \Omega(x_1), \Theta(x_1) \right) \\ &\quad \text{in Sone, Ohwada \& Aoki [1989a],} \\ \left( a_4, Y_{a4}(\eta) \right) &= \left( a_4, Y_{a4}(\eta) \right) \text{ in Ohwada \& Sone [1992].} \end{aligned} \right\} \quad (3.47)$$

<sup>14</sup>As noted just after Eq.(2.5), the velocity of the boundary must satisfy the condition  $u_{wi}n_i = 0$  for the state to be steady.

<sup>15</sup>The slip coefficients and Knudsen-layer functions depend on the parameter  $U_0/k_B T_0$  (Section 1.9) except for a hard-sphere molecular gas and the BKW model. This kind of note will not be repeated.

<sup>16</sup>The curvature tensor  $\kappa_{ij}$  is related to the variation of the normal  $n_i$ :  $\kappa_{ij} = (\partial s_1 / \partial x_i)_0 \partial n_j / \partial s_1 + (\partial s_2 / \partial x_i)_0 \partial n_j / \partial s_2$ .

<sup>17</sup>The data for the Maxwell-type condition are found in Ohwada & Sone [1992] and Wakabayashi, Ohwada, & Golse [1996].

**Table 3.1.** Knudsen-layer functions for the Boltzmann equation for hard-sphere molecules under the diffuse reflection condition or under the complete condensation condition. The functions of the same symbol, i.e.,  $Y_0$ ,  $Y_1$ ,  $\Omega_1$ , and  $\Theta_1$ , are common to the diffuse reflection and the complete condensation.

$\eta$	$Y_0(\eta)$	$Y_1(\eta)/2$	$\Omega_1(\eta)$	$-\Theta_1(\eta)$	$H_A(\eta)$	$H_B(\eta)$	$Y_{a4}(\eta)$	$\Omega_4^*(\eta)$	$\Theta_4^*(\eta)$
0.0000	0.34522	0.44508	0.51641	0.73783	0.13384	1.38948	0.29625	0.37815	0.05206
0.02503	0.30466	0.41870	0.46886	0.67376	0.12660	1.28950	0.28278	0.33624	0.04783
0.05011	0.28008	0.40046	0.43974	0.63360	0.12161	1.22172	0.27284	0.31065	0.04490
0.10108	0.24424	0.37107	0.39661	0.57323	0.11350	1.11477	0.25613	0.27306	0.04026
0.15138	0.21806	0.34744	0.36451	0.52766	0.10689	1.03068	0.24218	0.24538	0.03662
0.20226	0.19691	0.32686	0.33809	0.48977	0.10107	0.95884	0.22971	0.22286	0.03352
0.30081	0.16525	0.29328	0.29753	0.43097	0.09143	0.84434	0.20878	0.18886	0.02861
0.41305	0.13848	0.26186	0.26200	0.37886	0.08226	0.74030	0.18860	0.15979	0.02420
0.60049	0.10657	0.21995	0.21757	0.31303	0.06979	0.60615	0.16086	0.12468	0.01862
0.76791	0.08628	0.19014	0.18766	0.26839	0.06076	0.51404	0.14059	0.10205	0.01487
0.99185	0.06653	0.15806	0.15673	0.22204	0.05089	0.41800	0.11827	0.07973	0.01107
1.17977	0.05426	0.13621	0.13625	0.19136	0.04409	0.35452	0.10280	0.06570	0.00865
1.40407	0.04310	0.11471	0.11642	0.16171	0.03731	0.29358	0.08733	0.05279	0.00641
1.66967	0.03327	0.09416	0.09766	0.13382	0.03078	0.23688	0.07233	0.04130	0.00443
1.81952	0.02891	0.08444	0.08879	0.12072	0.02767	0.21058	0.06515	0.03615	0.00356
1.98156	0.02493	0.07517	0.08032	0.10828	0.02468	0.18585	0.05826	0.03142	0.00277
2.54689	0.01523	0.05063	0.05759	0.07538	0.01672	0.12204	0.03977	0.01974	0.00094
2.99541	0.01052	0.03733	0.04484	0.05739	0.01237	0.08856	0.02959	0.01397	0.00015
3.50590	0.00702	0.02658	0.03410	0.04257	0.00883	0.06213	0.02125	0.00960	-0.00035
4.08156	0.00453	0.01825	0.02528	0.03076	0.00607	0.04209	0.01471	0.00642	-0.00060
6.21632	0.00099	0.00472	0.00883	0.00993	0.00157	0.01057	0.00390	0.00167	-0.00054
7.97384	0.00030	0.00160	0.00385	0.00414	0.00053	0.00355	0.00135	0.00062	-0.00031
9.96662	0.00008	0.00048	0.00154	0.00159	0.00016	0.00106	0.00042	0.00022	-0.00015
15.00000	0.00000	0.00002	0.00016	0.00016	0.00001	0.00005	0.00003	0.00002	-0.00002
20.77429	0.00000	0.00000	0.00001	0.00001	0.00000	0.00000	0.00000	0.00000	-0.00000
25.39373	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

The Knudsen-layer functions  $Y_0(\eta)$ ,  $Y_1(\eta)$ ,  $\Omega_1(\eta)$ ,  $\Theta_1(\eta)$ , and  $Y_{a4}(\eta)$  are tabulated in Table 3.1. For the BKW model under the diffuse reflection, the slip coefficients and Knudsen-layer functions are obtained up to the 2nd-order in Sone [1969, 1971]. The slip coefficients are

$$\left. \begin{aligned} k_0 &= -1.01619, & K_1 &= -0.38316, & d_1 &= 1.30272, \\ a_1 &= 0.76632, & a_2 &= 0.50000, & a_3 &= -0.26632, \\ a_4 &= 0.27922, & a_5 &= 0.26693, & a_6 &= -0.76644, \\ b_1 &= 0.11684, & b_2 &= 0.26693, & d_3 &= 0, \\ d_4 &= 0.11169, & d_5 &= 1.82181. \end{aligned} \right\} \quad (3.48)$$

The Knudsen-layer functions  $Y_{a1}$  etc. can be or are put in the following form:

$$\left. \begin{aligned} Y_{a1} &= -Y_1, & Y_{a2} &= -2\tilde{Y}_0, & Y_{a3} &= -Y_1 - k_0 Y_0, \\ Y_{a4} &= 0, & Y_{a5} &= Y_2 - \tilde{Y}_1, & Y_{a6} &= \frac{1}{2}Y_2 - (K_1 + \frac{1}{4})Y_0, \\ (\Omega_{d4}, \Theta_{d4}) &= (-\frac{1}{4}\Omega_4^*, -\frac{1}{4}\Theta_4^*), & (\Omega_{d5}, \Theta_{d5}) &= (\Omega_5, \Theta_5), \\ \Omega_{d3} &= \Theta_{d3} = 0, \end{aligned} \right\} \quad (3.49)$$

and the functions on the right hand sides are tabulated in Table 3.2.

The second term on the right hand side of the slip condition (3.43a) shows that a flow is induced over a wall with a temperature gradient along it. The

**Table 3.2.** Knudsen-layer functions for the BKW equation under the diffuse-reflection condition or under the complete-condensation condition. The functions of the same symbol, i.e.,  $Y_0$ ,  $Y_1$ ,  $\Omega_1$ , and  $\Theta_1$ , are common to the diffuse reflection and the complete condensation.

$\eta$	$Y_0$	$Y_1$	$Y_2$	$\tilde{Y}_0$	$\tilde{Y}_1$	$\int_{\eta}^{\infty} Y_0 d\eta_0$	$\int_{\eta}^{\infty} Y_1 d\eta_0$
0.00	0.30908	0.54777	0.92525	0.3907	1.1922	0.23368	0.53385
0.05	0.25727	0.48117	0.83415	0.3569	1.1004	0.21974	0.50837
0.10	0.22827	0.43961	0.77446	0.3350	1.0391	0.20764	0.48539
0.20	0.18878	0.37876	0.68370	0.3017	0.9439	0.18693	0.44465
0.40	0.14003	0.29679	0.55502	0.2536	0.8032	0.15449	0.37770
0.60	0.10929	0.24081	0.46264	0.2181	0.6970	0.12976	0.32425
0.80	0.08773	0.19932	0.39151	0.1898	0.6115	0.11017	0.28042
1.00	0.07175	0.16720	0.33475	0.1666	0.5403	0.09430	0.24390
1.20	0.05948	0.14167	0.28842	0.1471	0.4800	0.08123	0.21310
1.40	0.04982	0.12096	0.25000	0.1305	0.4281	0.07033	0.18691
1.60	0.04209	0.10394	0.21777	0.1162	0.3832	0.06117	0.16447
1.80	0.03580	0.08978	0.19048	0.1038	0.3439	0.05340	0.14514
2.00	0.03063	0.07790	0.16720	0.0930	0.3093	0.04677	0.12841
2.50	0.02118	0.05551	0.12224	0.0712	0.2393	0.03399	0.09543
3.00	0.01498	0.04028	0.09068	0.0551	0.1870	0.02506	0.07172
4.00	0.00787	0.02206	0.05153	0.0338	0.1164	0.01407	0.04157
5.00	0.00434	0.01256	0.03023	0.0212	0.0740	0.00815	0.02474
10.00	0.00032	0.00104	0.00277	0.0025	0.0092	0.00071	0.00236
$(\eta \ln \eta)^*$	0.39894	0.40540	0.50519	0.2027	0.5052	0	0

$\eta$	$\Omega_1$	$-\Theta_1$	$\Omega_4^*$	$\Theta_4^* \times 10$	$\int_{\eta}^{\infty} (\Omega_1 + \Theta_1) d\eta_0$	$\int_{\eta}^{\infty} (\Omega_4^* + \Theta_4^*) d\eta_0$
0.00	0.34771	0.44920	0.36303	0.3717	-0.11609	0.23886
0.05	0.29177	0.38521	0.28722	0.3771	-0.11123	0.22106
0.10	0.26121	0.34842	0.24683	0.3639	-0.10672	0.20592
0.20	0.21987	0.29703	0.19399	0.3328	-0.09852	0.18060
0.40	0.16877	0.23114	0.13269	0.2757	-0.08466	0.14249
0.60	0.13610	0.18770	0.09669	0.2297	-0.07331	0.11479
0.80	0.11276	0.15607	0.07296	0.1927	-0.06385	0.09377
1.00	0.09507	0.13181	0.05634	0.1627	-0.05587	0.07738
1.20	0.08118	0.11258	0.04424	0.1382	-0.04908	0.06438
1.40	0.06999	0.09700	0.03519	0.1180	-0.04325	0.05392
1.60	0.06080	0.08417	0.02828	0.1012	-0.03822	0.04542
1.80	0.05315	0.07345	0.02293	0.0871	-0.03387	0.03844
2.00	0.04671	0.06440	0.01872	0.0752	-0.03007	0.03267
2.50	0.03442	0.04717	0.01156	0.0528	-0.02254	0.02210
3.00	0.02588	0.03521	0.00735	0.0376	-0.01707	0.01523
4.00	0.01524	0.02042	0.00315	0.0198	-0.01004	0.00754
5.00	0.00932	0.01231	0.00143	0.0108	-0.00606	0.00389
10.00	0.00107	0.00134	0.00004	0.0007	-0.00063	0.00021
$(\eta \ln \eta)^*$	0.46451	0.47587	0.63047	-0.5610	0	0

$\eta$	$\Omega_5$	$\Theta_5$	$-\Omega_6$	$-\Theta_6$	$\Omega_7$	$\Theta_7$
0.00	0.3516	-0.5956	0.31622	0.21690	0.5166	0.3136
0.05	0.2853	-0.4931	0.26242	0.19447	0.4441	0.2837
0.10	0.2452	-0.4292	0.23040	0.17970	0.3999	0.2640
0.20	0.1883	-0.3368	0.18563	0.15738	0.3364	0.2340
0.40	0.1153	-0.2163	0.12971	0.12619	0.2529	0.1916
0.60	0.0686	-0.1383	0.09488	0.10417	0.1976	0.1611
0.80	0.0363	-0.0838	0.07112	0.08746	0.1577	0.1375
1.00	0.0130	-0.0444	0.05413	0.07428	0.1276	0.1186
1.20	-0.0040	-0.0154	0.04161	0.06363	0.1043	0.1030
1.40	-0.0166	0.0061	0.03220	0.05487	0.0859	0.0900
1.60	-0.0260	0.0222	0.02501	0.04758	0.0712	0.0791
1.80	-0.0328	0.0341	0.01947	0.04146	0.0592	0.0697
2.00	-0.0378	0.0428	0.01515	0.03626	0.0495	0.0617
2.50	-0.0444	0.0550	0.00799	0.02632	0.0320	0.0459
3.00	-0.0460	0.0587	0.00400	0.01943	0.0209	0.0346
4.00	-0.0420	0.0545	0.00052	0.01097	0.0091	0.0203
5.00	-0.0349	0.0452	-0.00050	0.00643	0.0039	0.0122
10.00	-0.0093	0.0116	-0.00027	0.00061	-0.0001	0.0012
$(\eta \ln \eta)^*$	0.4291	-0.6043	0.34843	0.11459	0.4646	0.1528

\* The Knudsen-layer functions generally have the singularity  $\eta \ln \eta$  at  $\eta = 0$ . Its coefficient is shown in this row.

flow is called thermal creep flow (Kennard [1938], Sone [1966, 1970], Ohwada, Sone & Aoki [1989a]), whose simple experimental demonstration is given in Sone [1991b] and Sone, Sawada & Hirano [1994]. The fifth term on the right-hand side of Eq. (3.44a) shows the existence of another type of flow, called thermal stress slip flow (Sone [1972]). These flows, not expected in the classical fluid dynamics and discussed in more detail in Section 3.4.1, are typical examples of flows induced by the effect of gas rarefaction and cause thermophoresis<sup>18</sup>, which is one of the interesting subjects in aerosol science.

The Knudsen-layer parts of the stress tensor  $P_{ij}$  and the heat-flow vector  $Q_i$  are given by

$$P_{ijK0} = 0, \quad (3.50)$$

$$P_{ijK1} = -\frac{3}{2}G_{kG0}n_k[\Omega_1(\eta) + \Theta_1(\eta)](\delta_{ij} - n_in_j), \quad (3.51)$$

$$P_{ijK2}n_it_j = \frac{3}{2}\left(\frac{\partial G_{iG0}}{\partial x_j}n_it_j + G_{iG0}\kappa_{ij}t_j\right)\int_{-\infty}^{\eta}[\Omega_1(\eta_0) + \Theta_1(\eta_0)]d\eta_0, \quad (3.52a)$$

$$P_{ijK2}n_in_j = -3\bar{\kappa}G_{iG0}n_i\int_{-\infty}^{\eta}[\Omega_1(\eta_0) + \Theta_1(\eta_0)]d\eta_0, \quad (3.52b)$$

$$Q_{iK0} = 0, \quad (3.53)$$

$$Q_{iK1}n_i = 0, \quad Q_{iK1}t_i = -S_{ijG0}t_in_jH_A(\eta) - G_{iG0}t_iH_B(\eta), \quad (3.54)$$

$$\begin{aligned} Q_{iK2}n_i = & -\frac{1}{2}\frac{\partial S_{ijG0}}{\partial x_k}n_in_jn_k\int_{-\infty}^{\eta}H_A(\eta_0)d\eta_0 \\ & -\left(\frac{\partial G_{iG0}}{\partial x_j}n_in_j + 2\bar{\kappa}G_{iG0}n_i\right)\int_{-\infty}^{\eta}H_B(\eta_0)d\eta_0, \end{aligned} \quad (3.55)$$

where the quantities with the subscript  $G$  are evaluated on the boundary, and the functions  $H_A(\eta)$  and  $H_B(\eta)$  for a hard-sphere molecular gas are shown in Table 3.1 (Ohwada, Sone & Aoki [1989a]). For the BKW model,

$$H_A(\eta) = \frac{1}{2}Y_0(\eta), \quad H_B(\eta) = \frac{1}{4}Y_1(\eta) + \frac{1}{\sqrt{\pi}}\int_0^{\infty}\exp\left(-y^2 - \frac{\eta}{y}\right)dy. \quad (3.56)$$

### B On an interface of a gas and its condensed phase with evaporation or condensation

The boundary condition for the Stokes set of equations and the Knudsen-layer correction for the macroscopic variables on an interface of a gas and its condensed phase are given as follows (Sone & Onishi [1978], Sone & Aoki [1987],

<sup>18</sup>Thermophoresis is a phenomenon that a body in a rarefied gas (or a small particle in a gas) with a temperature gradient is subject to a force and drifts in the gas (see, for example, Chapter 4, Sone & Aoki [1977a, b, 1979, 1981, 1983], Ohwada & Sone [1992], Takata & Sone [1995]).



Sone [1991]):

$$(u_{iG0} - u_{wi0})t_i = 0, \quad (3.57a)$$

$$u_{iK0} = 0, \quad (3.57b)$$

$$\begin{bmatrix} P_{G0} - P_{w0} \\ \tau_{G0} - \tau_{w0} \\ \omega_{K0} \\ \tau_{K0} \end{bmatrix} = u_{iG0}n_i \begin{bmatrix} C_4^* \\ d_4^* \\ \Omega_4^*(\eta) \\ \Theta_4^*(\eta) \end{bmatrix}, \quad (3.57c)$$

$$\begin{bmatrix} (u_{iG1} - u_{wi1})t_i \\ u_{iK1}t_i \end{bmatrix} = S_{ijG0}n_it_j \begin{bmatrix} k_0 \\ Y_0(\eta) \end{bmatrix} + G_{iG0}t_i \begin{bmatrix} K_1 \\ \frac{1}{2}Y_1(\eta) \end{bmatrix} \\ + t_j \frac{\partial}{\partial x_j}(u_{iG0}n_i) \begin{bmatrix} K_2 \\ Y_{K2}(\eta) \end{bmatrix}, \quad (3.58a)$$

$$\begin{bmatrix} P_{G1} - P_{w1} \\ \tau_{G1} - \tau_{w1} \\ \omega_{K1} \\ \tau_{K1} \end{bmatrix} = u_{iG1}n_i \begin{bmatrix} C_4^* \\ d_4^* \\ \Omega_4^*(\eta) \\ \Theta_4^*(\eta) \end{bmatrix} - G_{iG0}n_i \begin{bmatrix} C_1 \\ d_1 \\ \Omega_1(\eta) \\ \Theta_1(\eta) \end{bmatrix} \\ - S_{ijG0}n_in_j \begin{bmatrix} C_6 \\ d_6 \\ \Omega_{d6}(\eta) \\ \Theta_{d6}(\eta) \end{bmatrix} - 2\bar{\kappa}u_{iG0}n_i \begin{bmatrix} C_7 \\ d_7 \\ \Omega_{d7}(\eta) \\ \Theta_{d7}(\eta) \end{bmatrix}, \quad (3.58b)$$

$$u_{iK1}n_i = 0, \quad (3.58c)$$

$$P_w = P_{w0} + P_{w1}k + \cdots, \quad (3.59)$$

where  $(2RT_0)^{1/2}u_{wi}$  ( $u_{wi}n_i = 0$ ) and  $T_0(1+\tau_w)$  are the velocity and temperature of the condensed phase, respectively;  $p_0(1+P_w)$  is the saturation gas pressure at temperature  $T_0(1+\tau_w)$ ; the quantities with the subscript  $G$  are evaluated on the interface; and  $k_0$ ,  $K_1$ ,  $d_1$ ,  $Y_0$ ,  $Y_1$ ,  $\Omega_1$ , and  $\Theta_1$  under the complete-condensation condition are common with those in Eqs. (3.43a)–(3.43c) under the diffuse-reflection condition.

For a hard-sphere molecular gas under the complete-condensation condition,

$$C_4^* = -2.1412, \quad d_4^* = -0.4557, \quad C_1 = 1.0947, \quad (3.60)$$

and  $(C_4^*, d_4^*, \eta, \Omega_4^*(\eta), \Theta_4^*(\eta)) = (\gamma + \delta, \delta, x_1, \Omega(x_1), \Theta(x_1))$  in Sone, Ohwada & Aoki [1989b].<sup>19</sup> The Knudsen-layer functions  $\Omega_4^*(\eta)$  and  $\Theta_4^*(\eta)$  are tabulated in

<sup>19</sup>The slip coefficients, corresponding to  $k_0$ ,  $K_1$ ,  $d_1$ ,  $C_1$ , and  $d_4^*$ , obtained in Loyalka [1989a, b, 1991] are not literally those for hard-sphere molecules. In their analysis, the kernel of the collision integral  $\mathcal{L}(\phi)$  [Eqs. (1.49)–(1.50c)] is expanded in the associated Legendre functions and truncated with two terms (and modified further) in Loyalka & Hickey [1989a] and with four terms in Loyalka [1989b] (not clear in Loyalka [1991]). It is not shown there how close the

Table 3.1. For BKW model under the complete-condensation condition, the slip (or jump) coefficients and Knudsen-layer functions are obtained up to the 1st order in Sone & Onishi [1973, 1978]. The slip coefficients are

$$\left. \begin{aligned} C_1 &= 0.55844, C_4^* = -2.13204, C_6 = 0.82085, C_7 = -0.38057, \\ K_2 &= -0.79519, d_4^* = -0.44675, d_6 = 0.33034, d_7 = -0.13157. \end{aligned} \right\} \quad (3.61)$$

The newly appeared Knudsen-layer functions  $Y_{K2}$  etc. can be or are put as follows:

$$Y_{K2} = 2Y_0 + \frac{1}{2}d_4^*Y_1, \quad (\Omega_{d6}, \Theta_{d6}) = (\Omega_6, \Theta_6), \quad (\Omega_{d7}, \Theta_{d7}) = (\Omega_7, \Theta_7), \quad (3.62)$$

and the functions on the right hand sides are tabulated in Table 3.2.

The Knudsen-layer parts of the stress tensor  $P_{ij}$  and the heat-flow vector  $Q_i$  are given by

$$P_{ijK0} = \frac{3}{2}u_{kG0}n_k[\Omega_4^*(\eta) + \Theta_4^*(\eta)](\delta_{ij} - n_in_j), \quad (3.63)$$

$$P_{ijK1}n_in_j = 3\bar{\kappa}u_{iG0}n_i \int_{-\infty}^{\infty} [\Omega_4^*(\eta_0) + \Theta_4^*(\eta_0)]d\eta_0, \quad (3.64a)$$

$$P_{ijK1}n_it_j = -\frac{3}{2}\frac{\partial u_{iG0}n_i}{\partial x_j}t_j \int_{-\infty}^{\infty} [\Omega_4^*(\eta_0) + \Theta_4^*(\eta_0)]d\eta_0, \quad (3.64b)$$

$$Q_{iK0} = 0, \quad (3.65)$$

$$Q_{iK1}n_i = 0, \quad (3.66)$$

where the quantities with the subscript  $G$  are evaluated on the boundary.

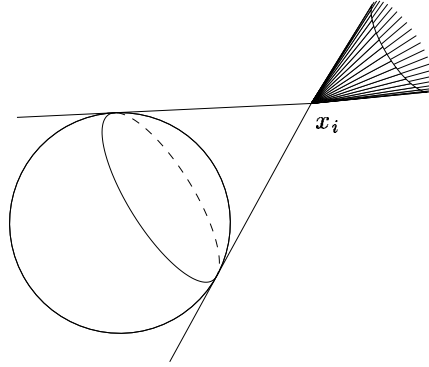
### 3.1.6 Discontinuity of the velocity distribution function and S layer

The Boltzmann equation (3.1) determines the variation of the velocity distribution function  $\phi$  only in the  $\zeta_i$  direction (or along the characteristic of the equation) in  $x_i$  space. Thus, if there is a discontinuity of  $\phi$  at  $(x_i = x_i^{(0)}, \zeta_i = \zeta_i^{(0)})$ , then the discontinuity propagates from  $x_i^{(0)}$  in the direction of  $\zeta_i^{(0)}$ .<sup>20</sup> This discontinuity decays owing to molecular collisions over the distance of the order of

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approximated kernel is to the exact one. According to Wakabayashi, Ohwada & Golse [1996] and H. Chihara (private communication), there is considerable difference between the exact kernel and its four term approximation. Thus, the solutions there are those of model equations, as the BKW equation, different from the Boltzmann equation for hard-sphere molecules. In fact, the Knudsen-layer profile in Loyalka & Hickey [1989a] differs considerably from that of Ohwada, Sone & Aoki [1989a], although their slip coefficients are close to those in Eqs. (3.46) and (3.60).

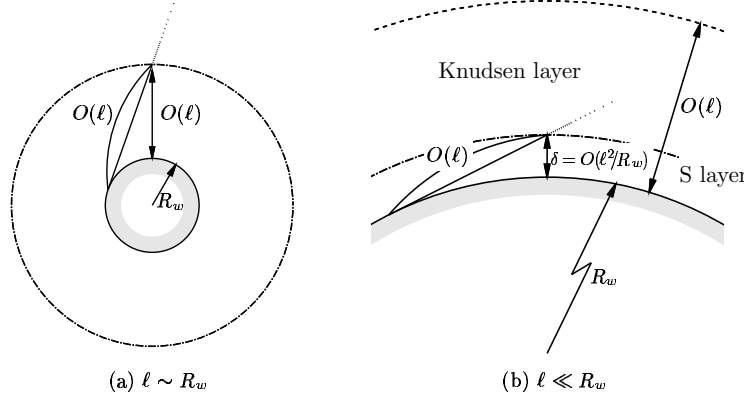
<sup>20</sup>The discontinuity of the velocity distribution function is not particular to the linearized or steady problem (see Chapters 4 and 5). Examples of the discontinuity in time-dependent problems are given in Sone & Sugimoto [1990] and Aoki, Sone, Nishino & Sugimoto [1991].



**Figure 3.1.** Discontinuity of the velocity distribution function in a gas around a convex boundary. At point  $x_i$ , the velocity distribution function is discontinuous on the shaded cone in  $\zeta_i$  space.

the free path of the molecules with  $\zeta_i = \zeta_i^{(0)}$ . From this property, we find that the velocity distribution function has discontinuity in a gas around a boundary with convex part [e.g., around a closed (or bounded) body], irrespective of the continuity of the boundary data, by the following reason. The velocity distribution of the molecules ( $\zeta_i n_i > 0$ ) leaving the boundary, which is determined by the boundary condition (3.2), is different in nature from that of the impinging molecules ( $\zeta_i n_i < 0$ ), which is formed by the interaction with the surrounding molecules. Thus, the velocity distribution function generally has discontinuity at the velocity ( $\zeta_i n_i = 0$ ) tangent to the boundary. This discontinuity propagates into the gas on the convex part of the boundary (Fig. 3.1). On a concave part of the boundary, on the other hand, the characteristic does not enter the gas region, and therefore the discontinuity does not propagate into the gas (Sone & Takata [1992]). The discontinuity of the velocity distribution function around a convex body is analyzed by detailed numerical computation in drag and thermal force problems around a sphere and in a strong evaporation problem from a spherical or cylindrical condensed phase (Takata, Sone & Aoki [1993], Sone, Takata & Wakabayashi [1994], Takata & Sone [1995], Sugimoto & Sone [1992], Sone & Sugimoto [1993], and Sone & Sugimoto [1995]).

When the mean free path  $\ell$  of the gas molecules is comparable to or larger than the radius  $R_w$  of the curvature of a convex body, the discontinuity is appreciable in the region where the distance from the boundary is of the order of the mean free path [Fig. 3.2 (a)]. This is the transition region where the gas molecules leaving the boundary are accommodated to the state of the surrounding gas by molecular collisions. That is, if  $\ell/R_w$  (Knudsen number based on the radius curvature) is not small, the region with the discontinuity extends in the same area as the transition region. On the other hand, if  $\ell/R_w$  is small, the discontinuity line is almost parallel to the boundary at the distance of the order of several  $\ell$  along the discontinuity, where the discontinuity has almost vanished. Thus, the discontinuity extends only in the region within  $O(\ell^2/R_w)$



**Figure 3.2.** Transition region and the region where the discontinuity of the velocity distribution function is appreciable. The discontinuity is tangent to the boundary and at the velocity of the molecules leaving the boundary along this line. The discontinuity decays with distance from the boundary owing to molecular collisions; it is appreciable on the solid line with length  $O(\ell)$ , and negligible farther away on the vanishing dotted line. For  $\ell \sim R_w$ , the discontinuity extends to a region, shown by the chain line, of  $O(\ell)$  from the boundary. For small  $\ell/R_w$ , the discontinuity is in a thin layer, shown by the chain line, with thickness  $\delta = O(\ell^2/R_w)$ ; the region, bounded by the dashed line, of  $O(\ell)$  from the boundary is the Knudsen layer.

from the boundary [Fig. 3.2 (b)]. The discontinuity exists only at the bottom of the transition region (called Knudsen layer for small Knudsen numbers).

The discontinuity of the velocity distribution function requires corrections to the result of the asymptotic analysis. That is, at the order next to that where the Knudsen-layer correction first appears (at  $k^2$  order for the diffuse reflection or at  $k$  order for the complete-condensation condition) a correction is required in a thin layer with thickness of the order of  $\ell^2/R_w$  at the bottom of the Knudsen layer. However, the corrections to the slip condition and the variables  $u_i n_i$ ,  $P_{ij} n_j$ , and  $Q_i n_i$  are at the higher order, and therefore the formulas for these variables given in this section do not need any correction. This second boundary layer at the bottom of the Knudsen layer is called S layer, which is found in the analysis of a flow along a circular cylinder with a temperature gradient along it by Sone [1973]. The numerical demonstration of the S layer is given in Sone & Takata [1992], where the correction of  $u_i n_i$  is shown to be of the higher order in contrast to other variables.

Now it is clear that there is generally a discontinuity of the velocity distribution function in a gas. Thus, the proof by Arkeryd [1972] that the equality in Eq. (1.33) holds only when the velocity distribution function is Maxwellian for a wide class of functions where discontinuous functions are included is essential to the statement of the H theorem. This is an example showing that theorems apparently only of mathematical interest are very important in physical statements.

### 3.1.7 Force and mass and energy transfers on a closed body

Consider the total force, total moment of force, and total mass and energy transfers on a closed (or bounded) body in a gas. These global quantities can be obtained only by the knowledge of the fluid-dynamic part as follows (Sone [1984c]). The body may be the condensed phase of the gas; there may be other bodies in the gas, or the domain may be enclosed by a boundary.

**[Proposition 3.1.1]** Force  $F_i$  and the moment  $MO_i$  of force (around the origin) acting on a closed body in a gas are expressed by the fluid-dynamic part  $P_{ijG}$  of the stress tensor as follows:

$$F_i = -p_0 L^2 \int_S P_{ijG} \hat{n}_j dS, \quad (3.67)$$

$$MO_i = -p_0 L^3 \int_S \epsilon_{ilk} x_l P_{kjG} \hat{n}_j dS, \quad (3.68)$$

where the surface  $S$  of integration is a closed surface enclosing only the body under consideration,  $\hat{n}_i$  is the outward unit normal vector to the surface  $S$ ,  $dS$  is the surface element (in the nondimensional  $x_i$  space) for integration, and  $\epsilon_{ijk}$  is Eddington's epsilon.

**[Proposition 3.1.2]** Mass  $MF$  and energy  $ET$  transferred to a closed body in a gas per unit time are expressed by the fluid-dynamic parts  $u_{iG}$  and  $Q_{iG}$  of the (nondimensional) flow velocity and heat-flow vector as follows:

$$MF = -\rho_0 (2RT_0)^{1/2} L^2 \int_S u_{iG} \hat{n}_i dS, \quad (3.69)$$

$$ET = -p_0 (2RT_0)^{1/2} L^2 \int_S \left( Q_{iG} + \frac{5}{2} u_{iG} \right) \hat{n}_i dS, \quad (3.70)$$

where the definitions of  $S$ ,  $dS$ , and  $\hat{n}_i$  are the same as in Proposition 3.1.1.

As the contribution of the thermal stress and “pressure stress” on the force and the moment of force, we have the following proposition.

**[Proposition 3.1.3]** The thermal stress  $\partial^2 \tau_G / \partial x_i \partial x_j$  and the “pressure stress”  $\partial^2 P_G / \partial x_i \partial x_j$  in the stress tensor  $P_{ijG}$  do not contribute to the force and the moment of force acting on a closed body.

### 3.1.8 Summary

Solving the Stokes set of equations (3.12)–(3.13c) under the slip boundary conditions on a simple boundary or on an interface given in Section 3.1.5 from the zero-th order, we obtain the fluid-dynamic parts of the density, flow velocity, and temperature of the gas and, on the basis of these information, their Knudsen-layer corrections from the formulas there. Further, we can obtain the stress tensor and heat-flow vector with the aid of Eqs. (3.24), (3.25), and (3.50)–(3.55) or (3.63)–(3.66). Thus the problem for the steady behavior of a slightly rarefied gas is reduced to solving the Stokes set of equations under the slip boundary

condition. That is, a problem for a slightly rarefied gas can be treated with the same easiness as the corresponding classical fluid-dynamic problem.

In the linearized problem, which we have treated in this section, the fluid-dynamic-type equations (i.e., the Stokes set of equations) remain the same form when we advance the degree of approximation. Thus, the effect of gas rarefaction for the macroscopic variables, such as density, flow velocity, and temperature, enters only through the slip boundary condition. The thermal creep flow and thermal stress slip flow, which appear, respectively, in the first- and the second-order slip conditions, are flows particular to a rarefied gas and will be discussed in more detail in Section 3.4.1. The boundary conditions in Eqs. (3.57a) and (3.57c) are those to be used in the continuum gas dynamics (or classical gas dynamics) on an interface of a gas and its condensed phase.

### 3.1.9 Supplement—viscosity and thermal conductivity

Up to now we have discussed the problem in nondimensional variables, and we have not given the formulas of viscosity and thermal conductivity, though we have mentioned of them. They are simply obtained by rewriting the formulas for the stress tensor and heat-flow vector in the dimensional variables. That is, the viscosity  $\mu$  and the thermal conductivity  $\lambda$  are expressed by the mean free path as follows:

$$\mu = \frac{\sqrt{\pi}}{2} \gamma_1 p_0 (2RT_0)^{-1/2} \ell_0, \quad (3.71)$$

$$\lambda = \frac{5\sqrt{\pi}}{4} \gamma_2 R p_0 (2RT_0)^{-1/2} \ell_0. \quad (3.72)$$

Thus, Knudsen number  $Kn$ , Reynolds number  $Re$ , and Mach number  $Ma$ , the latter two of which are defined by  $Re = UL\rho_0\mu^{-1}$  and  $Ma = U(\gamma RT_0)^{-1/2}$  [ $U$ : the characteristic flow velocity in the system,  $\gamma$ : the specific-heat ratio (5/3 for a monatomic gas)] and are important parameters in the classical gas dynamics, are not independent:

$$Ma = \left( \frac{3\pi}{40} \right)^{1/2} \gamma_1 Re Kn. \quad (3.73)$$

In the linearized theory, we neglected the nonlinear terms of the Mach number, but retained the quantities of the order of  $MaKn^n$ . This means that  $Ma \ll Kn$  or  $Re \ll Kn$ . The Stokes set of equations in Section 3.1.2 reflects this situation.

## 3.2 Weakly nonlinear theory

### 3.2.1 Problem

According to the discussion in Section 3.1.9, the asymptotic theory of the linearized Boltzmann equation developed in Section 3.1 is applicable only to the

case where the Reynolds number is very small. In the present section, we extend the asymptotic theory, according to Sone [1971] and Sone & Aoki [1987], so as to be applicable to the case where the Reynolds number takes a finite value.

When the Reynolds number is finite, the Mach number is of the same order as the Knudsen number from Eq. (3.73). This fact is introduced in the analysis in the following ways: the Mach number is one of the scales that indicate the deviation of the system from a uniform equilibrium state at rest, and this is extended to the velocity distribution function. That is, in this section, we consider the case where the deviation of the velocity distribution function from a uniform equilibrium state at rest is of the order of the Knudsen number. By this assumption, we extend the range of the velocity distribution function a little beyond that of the linearized theory. In terms of the macroscopic parameters, this is not only the condition on the Mach number but also the condition that the temperature variation on the boundary is of the order of the Knudsen number. Elimination of the latter condition is discussed in Section 3.3.1.B, from which a very interesting fact will come out.

Under the condition mentioned above, we investigate the asymptotic behavior for small Knudsen numbers of the time-independent boundary-value problem of Boltzmann equation. We cannot neglect the nonlinear terms of the (perturbed) velocity distribution function  $\phi$  when powers of  $k$  are considered, because  $\phi = O(k)$  by the assumption. Thus, the basic equation is the (nonlinear) Boltzmann equation (1.42) without  $\partial/\partial t$  term:

$$\zeta_i \frac{\partial \phi}{\partial x_i} = \frac{1}{k} [\mathcal{L}(\phi) + \hat{J}(\phi, \phi)]. \quad (3.74)$$

The boundary condition is written in an abstract form:

$$\phi = \phi_w, \quad (\zeta_i n_i > 0). \quad (3.75)$$

### 3.2.2 S expansion and fluid-dynamic-type equation

As in Section 3.1.2, we first look for a moderately varying solution of Eq. (3.74), whose length scale of variation is the order of the characteristic length  $L$  of the system  $[\partial \phi / \partial x_i = O(\phi)]$ , in a power series of  $k$  (S expansion):

$$\phi_S = \phi_{S1}k + \phi_{S2}k^2 + \cdots, \quad (3.76)$$

where the series starts from the 1st order term of  $k$  because  $\phi = O(k)$ , and  $\phi_{Sm} = O(1)$  in contrast to  $\phi_{Gm}$ . The subscript  $S$  is attached to show the situation explicitly. Corresponding to the expansion (3.76), the macroscopic variables  $\omega$ ,  $u_i$ ,  $\tau$ , etc. are also expanded in  $k$ :

$$h_S = h_{S1}k + h_{S2}k^2 + \cdots, \quad (3.77)$$

where  $h = \omega$ ,  $u_i$ ,  $\cdots$ . The relation between  $h_{Sm}$  and  $\phi_{Sm}$  is obtained by expanding Eq. (1.48), but their relation is a little complicated owing to the

nonlinearity, in contrast to that of  $h_{Gm}$  and  $\phi_{Gm}$ . For example,

$$\omega_{S1} = \int \phi_{S1} E \, d\zeta, \quad (3.78a)$$

$$u_{iS1} = \int \zeta_i \phi_{S1} E \, d\zeta, \quad (3.78b)$$

$$\frac{3}{2}\tau_{S1} = \int \left( \zeta_i^2 - \frac{3}{2} \right) \phi_{S1} E \, d\zeta, \quad (3.78c)$$

$$P_{S1} = \omega_{S1} + \tau_{S1}, \quad (3.78d)$$

$$\omega_{S2} = \int \phi_{S2} E \, d\zeta, \quad (3.79a)$$

$$u_{iS2} = \int \zeta_i \phi_{S2} E \, d\zeta - \omega_{S1} u_{iS1}, \quad (3.79b)$$

$$\frac{3}{2}\tau_{S2} = \int \left( \zeta_i^2 - \frac{3}{2} \right) \phi_{S2} E \, d\zeta - u_{iS1}^2 - \frac{3}{2}\omega_{S1}\tau_{S1}, \quad (3.79c)$$

$$P_{S2} = \omega_{S2} + \tau_{S2} + \omega_{S1}\tau_{S1}. \quad (3.79d)$$

Substituting the series (3.76) into Eq. (3.74) and arranging the same order terms of  $k$ , we obtain a series of linear integral equations for  $\phi_{Sm}$ :

$$\mathcal{L}(\phi_{S1}) = 0, \quad (3.80)$$

$$\mathcal{L}(\phi_{Sm}) = \zeta_i \frac{\partial \phi_{Sm-1}}{\partial x_i} - \sum_{r=1}^{m-1} \hat{J}(\phi_{Sm-r}, \phi_{Sr}), \quad (m = 2, 3, \dots). \quad (3.81)$$

These are the same type of equation as Eqs. (3.5) and (3.6) in the linear theory; the difference is the  $\hat{J}$  terms in the inhomogeneous term. In view of (3.78a)–(3.78c),

$$\phi_{S1} = \omega_{S1} + 2\zeta_i u_{iS1} + \left( \zeta_i^2 - \frac{3}{2} \right) \tau_{S1}. \quad (3.82)$$

The solvability condition of Eq. (3.81), corresponding to Eq. (3.8) in the linear theory, is

$$\int g \left( \zeta_i \frac{\partial \phi_{Sm-1}}{\partial x_i} - \sum_{r=1}^{m-1} \hat{J}(\phi_{Sm-r}, \phi_{Sr}) \right) E \, d\zeta = 0, \quad (3.83a)$$

where

$$g = 1, \quad \zeta_j, \quad \text{or} \quad \zeta_j^2.$$

The  $\Sigma$  term in Eq. (3.83a) can be shown to vanish. Thus,

$$\int g \zeta_k \frac{\partial \phi_{Sm-1}}{\partial x_k} E \, d\zeta = 0, \quad (m = 2, 3, \dots). \quad (3.83b)$$



Then, the solution of Eq. (3.81) is uniquely expressed in the following form:

$$\begin{aligned} \phi_{Sm} = & \hat{\phi}_{Sm} + \omega_{Sm} + 2\zeta_i \left( u_{iSm} + \sum_{r=1}^{m-1} \omega_{Sr} u_{iSm-r} \right) \\ & + \left( \zeta_j^2 - \frac{3}{2} \right) \left\{ \tau_{Sm} + \sum_{r=1}^{m-1} \left[ \omega_{Sr} \tau_{Sm-r} \right. \right. \\ & \left. \left. + \frac{2}{3} \left( u_{iSr} u_{iSm-r} + \sum_{h=1}^{m-1-r} \omega_{Sr} u_{iSh} u_{iSm-r-h} \right) \right] \right\}, \quad (3.84) \end{aligned}$$

where  $\hat{\phi}_{Sm}$  is the particular solution of Eq. (3.81) orthogonal to  $g$ :

$$\int g \hat{\phi}_{Sm} E \, d\mathbf{\zeta} = 0. \quad (3.85)$$

From the solvability condition (3.83b), we obtain the following Navier-Stokes-type set of partial differential equations for  $\omega_{Sm}$ ,  $u_{iSm}$ , and  $\tau_{Sm}$ :

$$\frac{\partial P_{S1}}{\partial x_i} = 0, \quad (3.86)$$

$$\frac{\partial u_{iS1}}{\partial x_i} = 0, \quad (3.87a)$$

$$u_{jS1} \frac{\partial u_{iS1}}{\partial x_j} = -\frac{1}{2} \frac{\partial P_{S2}}{\partial x_i} + \frac{1}{2} \gamma_1 \frac{\partial^2 u_{iS1}}{\partial x_j^2}, \quad (3.87b)$$

$$u_{jS1} \frac{\partial \tau_{S1}}{\partial x_j} = \frac{1}{2} \gamma_2 \frac{\partial^2 \tau_{S1}}{\partial x_j^2}, \quad (3.87c)$$

$$\frac{\partial u_{jS2}}{\partial x_j} = -u_{jS1} \frac{\partial \omega_{S1}}{\partial x_j}, \quad (3.88a)$$

$$\begin{aligned} u_{jS1} \frac{\partial u_{iS2}}{\partial x_j} + (\omega_{S1} u_{jS1} + u_{jS2}) \frac{\partial u_{iS1}}{\partial x_j} \\ = -\frac{1}{2} \frac{\partial}{\partial x_i} \left( P_{S3} - \frac{1}{6} (\gamma_1 \gamma_2 - 4\gamma_3) \frac{\partial^2 \tau_{S1}}{\partial x_j^2} \right) \\ + \frac{1}{2} \gamma_1 \frac{\partial^2 u_{iS2}}{\partial x_j^2} + \frac{1}{2} \gamma_4 \frac{\partial}{\partial x_j} \left[ \tau_{S1} \left( \frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i} \right) \right], \quad (3.88b) \end{aligned}$$

$$\begin{aligned} u_{jS1} \frac{\partial \tau_{S2}}{\partial x_j} + (\omega_{S1} u_{jS1} + u_{jS2}) \frac{\partial \tau_{S1}}{\partial x_j} - \frac{2}{5} u_{jS1} \frac{\partial P_{S2}}{\partial x_j} \\ = \frac{1}{5} \gamma_1 \left( \frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i} \right)^2 + \frac{1}{2} \frac{\partial^2}{\partial x_j^2} \left( \gamma_2 \tau_{S2} + \frac{1}{2} \gamma_5 \tau_{S1}^2 \right), \quad (3.88c) \end{aligned}$$

where  $\gamma_n$  is a constant related to the collision integral [ $\gamma_1$ ,  $\gamma_2$ ,  $\gamma_3$ , and  $\gamma_6$  are defined in Eq. (3.27);  $\gamma_4$  and  $\gamma_5$  will be defined in Eq. (3.96)] (see Footnote 5). In deriving those equations, as in the linear theory, we make use of the property that  $\phi_{Sm}$  takes a special form as a function  $\zeta_i$  owing to the isotropic property of the collision operators  $\mathcal{L}$  and  $\hat{J}$ . The order of this differential system remains unchanged if the level of approximation is advanced, in contrast to the Chapman–Enskog expansion (Chapman & Cowling [1952]).<sup>21</sup>

The velocity distribution function  $\phi_{Sm}$  is given in the following form:

$$\phi_{S1} = \phi_{eS1}, \quad (3.89)$$

$$\phi_{S2} = \phi_{eS2} - \zeta_i \zeta_j B(\zeta) \frac{\partial u_{iS1}}{\partial x_j} - \zeta_i A(\zeta) \frac{\partial \tau_{S1}}{\partial x_i}, \quad (3.90)$$

where

$$\left. \begin{aligned} \phi_{eS1} &= \omega_{S1} + 2\zeta_i u_{iS1} + \left( \zeta_j^2 - \frac{3}{2} \right) \tau_{S1}, \\ \phi_{eS2} &= \omega_{S2} + 2\zeta_i u_{iS2} + \left( \zeta_j^2 - \frac{3}{2} \right) \tau_{S2} \\ &\quad + 2\zeta_i \omega_{S1} u_{iS1} + \left( \zeta_j^2 - \frac{3}{2} \right) \left( \frac{2}{3} u_{iS1}^2 + \omega_{S1} \tau_{S1} \right) \\ &\quad + 2 \left( \zeta_i \zeta_j - \frac{1}{3} \zeta_\ell^2 \delta_{ij} \right) u_{iS1} u_{jS1} + 2\zeta_i \left( \zeta_j^2 - \frac{5}{2} \right) u_{iS1} \tau_{S1} \\ &\quad + \left( \frac{1}{2} \zeta_i^2 \zeta_j^2 - \frac{5}{2} \zeta_i^2 + \frac{15}{8} \right) \tau_{S1}^2, \end{aligned} \right\} \quad (3.91)$$

and  $A(\zeta)$  and  $B(\zeta)$  are, respectively, defined by Eqs. (3.19) and (3.20). The  $\phi_{eS1}$  and  $\phi_{eS2}$  are the first two expansion coefficients of the local Maxwellian in the power series of  $k$ :

$$\begin{aligned} E(1 + \phi_{eS}) &= \frac{1 + \omega_S}{\pi^{3/2} (1 + \tau_S)^{3/2}} \exp \left( -\frac{(\zeta_i - u_{iS})^2}{1 + \tau_S} \right) \\ &= E(1 + \phi_{eS1} k + \phi_{eS2} k^2 + \dots). \end{aligned} \quad (3.92)$$

The coefficient functions of the S expansion of the stress tensor  $P_{ij}$  and heat-flow vector  $Q_i$  are given as follows:

$$\left. \begin{aligned} P_{ijS1} &= P_{S1} \delta_{ij}, \\ P_{ijS2} &= P_{S2} \delta_{ij} - \gamma_1 \left( \frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i} \right), \\ P_{ijS3} &= P_{S3} \delta_{ij} - \gamma_1 \left( \frac{\partial u_{iS2}}{\partial x_j} + \frac{\partial u_{jS2}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{kS2}}{\partial x_k} \delta_{ij} \right) \\ &\quad - \gamma_4 \tau_{S1} \left( \frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i} \right) + \gamma_3 \left( \frac{\partial^2 \tau_{S1}}{\partial x_i \partial x_j} - \frac{1}{3} \frac{\partial^2 \tau_{S1}}{\partial x_k^2} \delta_{ij} \right), \end{aligned} \right\} \quad (3.93)$$

<sup>21</sup>The structure of the Chapman–Enskog expansion is clearly and briefly explained in Grad [1958].

$$\left. \begin{aligned} Q_{iS1} &= 0, \\ Q_{iS2} &= -\frac{5}{4}\gamma_2 \frac{\partial \tau_{S1}}{\partial x_i}, \\ Q_{iS3} &= -\frac{5}{4}\gamma_2 \frac{\partial \tau_{S2}}{\partial x_i} - \frac{5}{4}\gamma_5 \tau_{S1} \frac{\partial \tau_{S1}}{\partial x_i} + \frac{1}{2}\gamma_3 \frac{\partial^2 u_{iS1}}{\partial x_j^2}, \end{aligned} \right\} \quad (3.94)$$

where  $\gamma_1$ ,  $\gamma_2$ , and  $\gamma_6$  are defined in Eq. (3.27), and  $\gamma_4$  and  $\gamma_5$  are defined as follows. Functions  $C(\zeta)$  and  $G(\zeta)$  of  $\zeta$  are first defined by the equations:

$$\left. \begin{aligned} 2\hat{J} \left( \zeta^2 - \frac{3}{2}, \zeta_i \zeta_j B(\zeta) \right) &= \zeta_i \zeta_j C(\zeta) + \delta_{ij} D(\zeta), \\ 2\hat{J} \left( \zeta^2 - \frac{3}{2}, \zeta_i A(\zeta) \right) &= \zeta_i G(\zeta). \end{aligned} \right\} \quad (3.95)$$

These forms of equations are compatible owing to the isotropic property of the collision operator  $\hat{J}$ . With  $A(\zeta)$ ,  $B(\zeta)$ ,  $C(\zeta)$ ,  $D(\zeta)$ , and  $G(\zeta)$ , the constants  $\gamma_4$  and  $\gamma_5$  are defined by the equations:

$$\left. \begin{aligned} \gamma_4 &= -\frac{5}{2}\gamma_1 + I_8(B) + \frac{1}{2}I_6(BC), \\ \gamma_5 &= -6\gamma_2 + 2I_8(A) + 2I_4(AG), \end{aligned} \right\} \quad (3.96)$$

where  $I_n$  is defined by Eq. (3.28). For a hard-sphere molecular gas,

$$\gamma_4 = 0.635021, \quad \gamma_5 = 0.961142, \quad (3.97)$$

and for the BKW model,

$$\gamma_4 = \gamma_5 = 1. \quad (3.98)$$

The last terms of  $P_{ijS3}$  and  $Q_{iS3}$ , i.e.,  $\gamma_3 [\partial^2 \tau_{S1} / \partial x_i \partial x_j - (1/3)(\partial^2 \tau_{S1} / \partial x_k^2) \delta_{ij}]$  and  $(\gamma_3/2) \partial^2 u_{iS1} / \partial x_j^2$ , are not found in Newton and Fourier formulas respectively. The former is called thermal stress. The terms before these terms ( $\gamma_4$  and  $\gamma_5$  terms) are due to the dependence of the viscosity and thermal conductivity on the temperature of the gas.

### 3.2.3 Knudsen layer and slip boundary condition

The S expansion  $\phi_S$ , as the Grad–Hilbert expansion  $\phi_G$ , does not have freedom enough to be fitted to the kinetic boundary condition (3.75), because each term  $\phi_{Sm}$  of the expansion is a special form of  $\zeta_i$ . Thus, we put the solution  $\phi$  of the boundary-value problem (3.74) and (3.75) in the form:

$$\phi = \phi_S + \phi_K, \quad (3.99)$$

where  $\phi_K$  is assumed to have a similar behavior  $[kn_i \partial \phi_K / \partial x_i = O(\phi_K)]$  to  $\phi_K$  in Section 3.1, that is,  $\phi_K$  is appreciable only in a thin layer, with thickness of the order of the mean free path, adjacent to the boundary and decays very rapidly [see Eq. (3.108) and its explanation] in the layer in the direction normal to the

boundary. The  $\phi_S$  is called the fluid-dynamic part, and  $\phi_K$  is the Knudsen layer part (or correction). The  $\phi_K$  is also expanded in the power series of  $k$ :

$$\phi_K = \phi_{K1}k + \phi_{K2}k^2 + \cdots, \quad (3.100)$$

where  $\phi_{Km} = O(1)$  in contrast to the same symbol  $\phi_{Km}$  in Section 3.1.

Here we introduce the Knudsen layer variables  $(\eta, s_1, s_2)$  [see Eq. (3.34) and its explanation]:

$$x_i = k\eta n_i(s_1, s_2) + x_{wi}(s_1, s_2). \quad (3.101)$$

Substituting Eq. (3.99) with the expansions (3.76) and (3.100) in Eq. (3.74) and rewriting them in the new variables, we obtain a series of one-dimensional (homogeneous or inhomogeneous) linearized Boltzmann equations for  $\phi_{Km}$ :

$$\zeta_i n_i \frac{\partial \phi_{K1}}{\partial \eta} = \mathcal{L}(\phi_{K1}), \quad (3.102)$$

$$\begin{aligned} \zeta_i n_i \frac{\partial \phi_{K2}}{\partial \eta} = & \mathcal{L}(\phi_{K2}) + 2\hat{J}((\phi_{S1})_0, \phi_{K1}) + \hat{J}(\phi_{K1}, \phi_{K1}) \\ & - \zeta_i \left[ \left( \frac{\partial s_1}{\partial x_i} \right)_0 \frac{\partial \phi_{K1}}{\partial s_1} + \left( \frac{\partial s_2}{\partial x_i} \right)_0 \frac{\partial \phi_{K1}}{\partial s_2} \right], \end{aligned} \quad (3.103)$$

where  $(\quad)_0$ , the parentheses with subscript 0, indicates that the quantities in the parentheses  $(\quad)$  are evaluated at  $\eta = 0$ . In these equations the terms that contain only  $\phi_S$  disappear because  $\phi_S$  satisfies Eq. (3.74), but the product terms of  $\phi_S$  and  $\phi_K$  remain. In the product terms, the series expansion of  $\phi_S$  with respect to  $\eta$ :

$$\begin{aligned} \phi_S &= (\phi_S)_0 + \left( \frac{\partial \phi_S}{\partial x_i} \right)_0 n_i k \eta + \cdots \\ &= (\phi_{S1})_0 k + \left[ (\phi_{S2})_0 + \left( \frac{\partial \phi_{S1}}{\partial x_i} \right)_0 n_i \eta \right] k^2 + \cdots, \end{aligned} \quad (3.104)$$

can be used owing to the assumption of fast decay of  $\phi_K$  with  $\eta$ . Thus, the terms relating to  $\phi_S$  are evaluated at  $\eta = 0$ . In the expansion,  $\eta$  always appears in the form  $k\eta$  and this leads to reshuffling the order of  $k$  in the expansion of  $\phi_S$ .

Corresponding to Eq. (3.99), the Knudsen layer parts ( $\omega_K, u_{iK}, \tau_K$ , etc.) of the macroscopic variables ( $\omega, u_i, \tau$ , etc.) are defined as the remainders ( $\omega_K = \omega - \omega_S, u_{iK} = u_i - u_{iS}, \tau_K = \tau - \tau_S$ , etc.). Then they depend on  $\phi_S$  as well as  $\phi_K$ , because the relations between the macroscopic variables and the velocity distribution function  $\phi$ , i.e., Eq. (1.48), are nonlinear. For example,

$$\left. \begin{aligned} u_{iK1} &= \int \zeta_i \phi_{K1} E \, d\zeta, \\ u_{iK2} &= \int \zeta_i \phi_{K2} E \, d\zeta - (\omega_{S1})_0 u_{iK1} - (u_{iS1})_0 \omega_{K1} - \omega_{K1} u_{iK1}. \end{aligned} \right\} \quad (3.105)$$

The boundary conditions for  $\phi_{Km}$  are as follows: at  $\eta = 0$

$$\phi_{Km} = \phi_{wm} - \phi_{Sm}, \quad (\zeta_i n_i > 0), \quad (3.106)$$

where  $\phi_{wm}$  is the expansion coefficient of  $\phi_w$  (see Footnote 6):

$$\phi_w = \phi_{w1}k + \phi_{w2}k^2 + \cdots, \quad (3.107)$$

and as  $\eta \rightarrow \infty$ ,

$$\phi_{Km} \rightarrow 0, \quad (3.108)$$

where the decay is assumed to be faster than any inverse power of  $\eta$ . This assumption is verified in Bardos, Caflisch & Nicolaenko [1986].

From analysis of the half space problem of the linearized (homogeneous or inhomogeneous) Boltzmann equation [Eq. (3.102) or (3.103) and Eqs. (3.106) and (3.108)], we obtain the slip boundary condition for the Navier–Stokes-type set of equations and the Knudsen-layer correction for the macroscopic variables by a similar process to that explained at the end of Section 3.1.4.

Here are summarized the slip boundary condition and the Knudsen-layer correction for the macroscopic variables on a simple boundary and on an interface of a gas and its condensed phase (Sone [1971, 1991a, c], Onishi & Sone [1979], Sone & Aoki [1987]). The boundary is assumed to be locally isotropic.<sup>22</sup>

**(a) On a simple solid boundary**

$$u_{iS1} - u_{wi1} = 0, \quad (3.109a)$$

$$\tau_{S1} - \tau_{w1} = 0, \quad (3.109b)$$

$$u_{iK1} = \omega_{K1} = \tau_{K1} = 0, \quad (3.109c)$$

$$\begin{bmatrix} (u_{iS2} - u_{wi2})t_i \\ u_{iK2}t_i \end{bmatrix} = S_{ijS1}n_it_j \begin{bmatrix} k_0 \\ Y_0(\eta) \end{bmatrix} + G_{iS1}t_i \begin{bmatrix} K_1 \\ \frac{1}{2}Y_1(\eta) \end{bmatrix}, \quad (3.110a)$$

$$\begin{bmatrix} u_{iS2}n_i \\ u_{iK}n_i \end{bmatrix} = 0, \quad (3.110b)$$

$$\begin{bmatrix} \tau_{S2} - \tau_{w2} \\ \omega_{K2} \\ \tau_{K2} \end{bmatrix} = -G_{iS1}n_i \begin{bmatrix} d_1 \\ \Omega_1(\eta) \\ \Theta_1(\eta) \end{bmatrix}, \quad (3.110c)$$

$$S_{ijSm} = -\left(\frac{\partial u_{iSm}}{\partial x_j} + \frac{\partial u_{jSm}}{\partial x_i}\right), \quad G_{iSm} = -\frac{\partial \tau_{Sm}}{\partial x_i},$$

where the slip coefficient  $k_0$ ,  $K_1$ , and  $d_1$  and the Knudsen-layer functions  $Y_0(\eta)$ ,  $Y_1(\eta)$ ,  $\Omega_1(\eta)$ , and  $\Theta_1(\eta)$  are the same as those in the linear theory, and the quantities with the subscript  $S$  are evaluated on the boundary.

The Knudsen layer parts of  $P_{ij}$  and  $Q_i$  are

$$P_{ijK1} = 0, \quad (3.111)$$

$$P_{ijK2} = -\frac{3}{2}G_{kS1}n_k[\Omega_1(\eta) + \Theta_1(\eta)](\delta_{ij} - n_in_j), \quad (3.112)$$

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<sup>22</sup>See Footnote 11

$$Q_{iK1} = 0, \quad (3.113)$$

$$Q_{iK2}n_i = 0, \quad Q_{iK2}t_i = -S_{ijS1}t_in_jH_A(\eta) - G_{iS1}t_iH_B(\eta), \quad (3.114)$$

where  $H_A(\eta)$  and  $H_B(\eta)$  are the same as those in the linear theory, and the quantities with the subscript  $S$  are evaluated on the boundary.

The slip boundary condition and the Knudsen layer correction up to the 2nd order of  $k$  are essentially the same as those (up to the 1st order in  $k$ ) in the linear theory.

### (b) On an interface of a gas and its condensed phase

$$(u_{iS1} - u_{wi1})t_i = 0, \quad (3.115a)$$

$$u_{iK1} = 0, \quad (3.115b)$$

$$\begin{bmatrix} P_{S1} - P_{w1} \\ \tau_{S1} - \tau_{w1} \\ \omega_{K1} \\ \tau_{K1} \end{bmatrix} = u_{iS1}n_i \begin{bmatrix} C_4^* \\ d_4^* \\ \Omega_4^*(\eta) \\ \Theta_4^*(\eta) \end{bmatrix}, \quad (3.115c)$$

$$\begin{bmatrix} (u_{iS2} - u_{wi2})t_i \\ u_{iK2}t_i \end{bmatrix} = S_{ijS1}n_it_j \begin{bmatrix} k_0 \\ Y_0 \end{bmatrix} + G_{iS1}t_i \begin{bmatrix} K_1 \\ \frac{1}{2}Y_1 \end{bmatrix} \\ + t_j \frac{\partial}{\partial x_j} (u_{iS1}n_i) \begin{bmatrix} K_2 \\ Y_{K2} \end{bmatrix}, \quad (3.116a)$$

$$u_{iK2}n_i = -(u_{iS1}n_i)^2\Omega_4^*, \quad (3.116b)$$

$$\begin{bmatrix} P_{S2} - P_{w2} \\ \tau_{S2} - \tau_{w2} \\ \omega_{K2} \\ \tau_{K2} \end{bmatrix} = u_{iS2}n_i \begin{bmatrix} C_4^* \\ d_4^* \\ \Omega_4^*(\eta) \\ \Theta_4^*(\eta) \end{bmatrix} - G_{iS1}n_i \begin{bmatrix} C_1 \\ d_1 \\ \Omega_1(\eta) \\ \Theta_1(\eta) \end{bmatrix} \\ - S_{ijS1}n_in_j \begin{bmatrix} C_6 \\ d_6 \\ \Omega_6(\eta) \\ \Theta_6(\eta) \end{bmatrix} - 2\bar{\kappa}u_{iS1}n_i \begin{bmatrix} C_7 \\ d_7 \\ \Omega_7(\eta) \\ \Theta_7(\eta) \end{bmatrix} \\ + (u_{iS1}n_i)^2 \begin{bmatrix} C_8 \\ d_8 \\ \Omega_8(\eta) \\ \Theta_8(\eta) \end{bmatrix} + \tau_{w1}u_{iS1}n_i \begin{bmatrix} C_9 \\ d_9 \\ \Omega_9(\eta) \\ \Theta_9(\eta) \end{bmatrix} \\ + P_{w1}u_{iS1}n_i \begin{bmatrix} C_{10} \\ d_{10} \\ \Omega_{10}(\eta) \\ \Theta_{10}(\eta) \end{bmatrix}, \quad (3.116c)$$

where the slip (jump) coefficients  $C_4^*$ ,  $d_4^*$ ,  $k_0$ ,  $K_1$ ,  $K_2$ ,  $C_6$ ,  $d_6$ ,  $C_7$ , and  $d_7$  and the Knudsen-layer functions  $\Omega_4^*(\eta)$ ,  $\Theta_4^*(\eta)$ ,  $\Omega_1(\eta)$ ,  $\Theta_1(\eta)$ ,  $\Omega_6(\eta)$ ,  $\Theta_6(\eta)$ ,  $\Omega_7(\eta)$ ,

and  $\Theta_7(\eta)$  are the same as those in the linear theory, and the quantities with the subscript  $S$  are evaluated on the interface. The leading term of the slip condition and the Knudsen layer correction are essentially the same as those in the linear theory.

For the BKW equation under the complete-condensation condition, the remaining coefficients  $C_8, d_8, C_9, d_9, C_{10}$ , and  $d_{10}$  and the Knudsen-layer functions  $\Omega_8(\eta), \Theta_8(\eta), \Omega_9(\eta), \Theta_9(\eta), \Omega_{10}(\eta)$ , and  $\Theta_{10}(\eta)$  are obtained (Onishi & Sone [1979]).<sup>23</sup> The slip coefficients are

$$\left. \begin{aligned} C_8 &= 2.320074, & C_9 &= 1.066019, & C_{10} &= C_4^*, \\ d_8 &= -0.0028315, & d_9 &= -0.223375, & d_{10} &= 0. \end{aligned} \right\} \quad (3.117)$$

The Knudsen-layer parts of the stress tensor  $P_{ij}$  and the heat-flow vector  $Q_i$  are given by

$$P_{ijK1} = \frac{3}{2} u_{kS1} n_k [\Omega_4^*(\eta) + \Theta_4^*(\eta)] (\delta_{ij} - n_i n_j), \quad (3.118)$$

$$P_{ijK2} n_i n_j = 3\bar{\kappa} u_{iS1} n_i \int_{-\infty}^{\eta} [\Omega_4^*(\eta_0) + \Theta_4^*(\eta_0)] d\eta_0, \quad (3.119a)$$

$$P_{ijK2} n_i t_j = -\frac{3}{2} t_j \frac{\partial u_{iS1} n_i}{\partial x_j} \int_{-\infty}^{\eta} [\Omega_4^*(\eta_0) + \Theta_4^*(\eta_0)] d\eta_0, \quad (3.119b)$$

$$Q_{iK1} = 0, \quad (3.120)$$

$$Q_{iK2} n_i = (u_{kS1} n_k)^2 \left( \Omega_4^*(\eta) - \frac{3}{2} \Theta_4^*(\eta) \right), \quad (3.121)$$

where  $u_{iS1} n_i$  is evaluated on the boundary.

### 3.2.4 Rarefaction effect of a gas

Equations (3.87a) and (3.87b) are of the same form as the mass and momentum-conservation equations of the Navier–Stokes set of equations for an incompressible fluid; Equation (3.87c) is slightly different from the corresponding energy equation.<sup>24</sup> The next order equations (3.88a)–(3.88c) [together with Eqs. (3.86), (3.87a)–(3.87c)] are very much like the Navier–Stokes set of equations for a

<sup>23</sup>In Onishi & Sone [1979],  $\int_0^\eta (1 + \omega) d\eta$  instead of  $\eta$  is used as a Knudsen-layer variable. The difference affects  $(\Omega_8, \Theta_8)$ ,  $(\Omega_9, \Theta_9)$ , and  $(\Omega_{10}, \Theta_{10})$ .

<sup>24</sup>Equation (3.87c) is a little different from the energy equation in the Navier–Stokes set of equations for an incompressible fluid [ $v_i \partial \rho / \partial X_i = 0$  (or  $u_i \partial \omega / \partial x_j = 0$ )] under the present situation with a small velocity ( $u_i \ll 1$  or small Mach number), a small temperature variation ( $\tau \ll 1$ ), and a finite Reynolds number. In the latter, the convection term  $u_{jS1} \partial \tau_{S1} / \partial x_j$  should be replaced by  $(3/5) u_{jS1} \partial \tau_{S1} / \partial x_j$ . This difference comes from the fact that the work done by pressure is of higher order in the incompressible fluid under the present situation. However, if we consider an incompressible fluid with the thermal conductivity multiplied by  $3/5$ , the set (3.86)–(3.87c) is the same as the incompressible Navier–Stokes set. For economy of words, we will, hereafter, call the set of equations “the Navier–Stokes set of equations for an incompressible fluid” or “the incompressible Navier–Stokes set of equations” with the quotation mark.

slightly compressible fluid, but there is a difference. Substituting the Mach number expansion of the macroscopic variables in the Navier–Stokes set of equations for a compressible fluid and transforming the Mach number expansion to the Knudsen number expansion in view of the fact that the case  $Ma = O(Kn)$  is being considered, then we successively obtain the equations (3.86), (3.87a)–(3.87c), and (3.88a)–(3.88c) with  $\gamma_3 = 0$ . The difference comes from the thermal stress in  $P_{ijS3}$  in Eq. (3.93). Introducing a new variable  $P_{S3}^*$ :

$$P_{S3}^* = P_{S3} + \frac{2}{3}\gamma_3 \frac{\partial^2 \tau_{S1}}{\partial x_i^2}, \quad (3.122)$$

we can incorporate the  $\gamma_3$  term in the pressure term. Thus, Eqs. (3.88a)–(3.88c) are apparently of the same form as the corresponding equations of the Mach number expansion on the basis of the Navier–Stokes set of equations. Further the slip conditions in Eqs. (3.109a)–(3.110c) or in Eq. (3.115a)–(3.116c) do not contain  $P_{S3}$ . Thus, we have the following proposition:

**[Proposition 3.2.1]** Except for the Knudsen-layer correction, the macroscopic variables of a slightly rarefied gas are obtained correctly up to the 2nd order of the Knudsen number (i.e., up to  $\omega_{S2}$ ,  $u_{iS2}$ ,  $\tau_{S2}$ , etc.) by solving the Navier–Stokes sets of equations for a slightly compressible fluid (or with the 1st order correction of compressibility) under the slip boundary conditions in Eqs. (3.109a)–(3.110c) or in Eqs. (3.115a)–(3.116c). The effect of rarefaction of the gas comes in only through the slip boundary conditions.

### 3.2.5 Force and mass and energy transfers on a closed body

We will give the extension of the propositions on the force and the moment of force or the mass and energy transfers on a closed body in the linear theory (Section 3.1.7) to the present weakly nonlinear case (Sone & Aoki [1987]).

Consider the total force, total moment of force, and total mass and energy transfers on a closed (or bounded) body in a gas. These global quantities can be obtained only by the knowledge of the fluid-dynamic part as follows. The body may be the condensed phase of the gas; there may be other bodies in the gas, or the domain may be enclosed by a boundary.

**[Proposition 3.2.2]** Force  $F_i$  and the moment  $MO_i$  of force (around the origin) acting on a closed body in a gas are expressed by the fluid-dynamic part  $\psi_{ijS}$  [=  $P_{ijS} + 2(1 + \omega_S)u_{iS}u_{jS}$ ] of the momentum flux tensor as follows:

$$F_i = -p_0 L^2 \int_S \psi_{ijS} \hat{n}_j dS, \quad (3.123)$$

$$MO_i = -p_0 L^3 \int_S \epsilon_{ilk} x_l \psi_{kjs} \hat{n}_j dS, \quad (3.124)$$

where the surface  $S$  of integration is a closed surface enclosing only the body under consideration,  $\hat{n}_i$  is the outward unit normal vector to the surface  $S$ ,  $dS$



is the surface element (in the nondimensional  $x_i$  space) for integration, and  $\epsilon_{ijk}$  is Eddington's epsilon.

**[Corollary]** On a simple boundary,  $F_i/p_0L^2$  and  $MO_i/p_0L^3$  are expressed only by  $P_{ijS}$  up to the third order of  $k$ .

**[Proposition 3.2.3]** Mass  $MF$  and energy  $ET$  transferred to a closed body in a gas per unit time are expressed by the fluid-dynamic parts  $\psi_{iS}^m [= (1 + \omega_S)u_{iS}]$  and  $\psi_{iS}^e [= \int \zeta_i \zeta_j^2 \phi_S E \, d\zeta = (5/2)u_{iS} + u_{jS}P_{ijS} + (3/2)P_S u_{iS} + (1 + \omega_S)u_{iS}u_{jS}^2 + Q_{iS}]$  of the (nondimensional) mass-flow vector and heat-flow vector as follows:

$$MF = -\rho_0(2RT_0)^{1/2}L^2 \int_S \psi_{iS}^m \hat{n}_i dS, \quad (3.125)$$

$$ET = -p_0(2RT_0)^{1/2}L^2 \int_S \psi_{iS}^e \hat{n}_i dS, \quad (3.126)$$

where the definitions of  $S$ ,  $dS$ , and  $\hat{n}_i$  are the same as in Proposition 3.2.2.

Consider the contribution of the thermal stress. First rewrite the expression of  $P_{ijS3}$  given by Eq. (3.93) using  $P_{S3}^*$  defined by Eq. (3.122) as follows:

$$\begin{aligned} P_{ijS3} = & P_{S3}^* \delta_{ij} - \gamma_1 \left( \frac{\partial u_{iS2}}{\partial x_j} + \frac{\partial u_{jS2}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{kS2}}{\partial x_k} \delta_{ij} \right) \\ & - \gamma_4 \tau_{S1} \left( \frac{\partial u_{iS1}}{\partial x_j} + \frac{\partial u_{jS1}}{\partial x_i} \right) + \gamma_3 \left( \frac{\partial^2 \tau_{S1}}{\partial x_i \partial x_j} - \frac{\partial^2 \tau_{S1}}{\partial x_k^2} \delta_{ij} \right). \end{aligned} \quad (3.127)$$

We will call this expression the stress in the  $P_{S3}^*$  system. In this system the thermal stress (non-Newtonian stress) may be taken as

$$\gamma_3 \left( \frac{\partial^2 \tau_{S1}}{\partial x_i \partial x_j} - \frac{\partial^2 \tau_{S1}}{\partial x_k^2} \delta_{ij} \right). \quad (3.128)$$

**[Proposition 3.2.4]** The non-Newtonian stress in the  $P_{S3}^*$  system contributes neither to the force nor to the moment of force on a closed body in a gas.

From Propositions 3.2.1 and 3.2.4, we have

**[Proposition 3.2.5]** Under the condition of Proposition 3.2.1, the force and the moment of force on a closed body can be obtained up to the third order of  $k$  in the expression  $F_i/p_0L^2$  and  $MO_i/p_0L^3$  by the classical gas dynamic calculation where the slip condition is taken into account. Here we mean by the classical gas dynamic calculation that the force or the moment of force is calculated with the aid of the Newtonian stress from the flow velocity and pressure fields obtained by solving the Navier–Stokes set of equations under the slip boundary condition.

### 3.2.6 Summary

In this section we discussed the asymptotic behavior of a gas for small Knudsen numbers in the case where the deviation from a uniform equilibrium state at rest was of the order of the Knudsen number. The systematic process of

derivation and the formulas (up to the second order of the Knudsen number) of the fluid-dynamic-type equations and their associate slip boundary conditions were shown. It should be noted that the fluid-dynamic-type equations that are to be used with the slip conditions (3.110a)–(3.110c) or (3.116a) and (3.116c) are not the Navier–Stokes set of equations but contain the thermal stress term. After the discussion of Section 3.2.4, it is found that the behavior of a gas up to this order can be treated by the Navier–Stokes set with the replacement (3.122). Incidentally, when the temperature variation in a gas is not small (Section 3.3.1.B), the thermal stress term cannot be included in the pressure term, and therefore the fluid-dynamic-type equations that should be used with the (leading) slip condition contain a term that is not found in the Navier–Stokes set of equations. This introduces an interesting result particular to a rarefied gas. [Moreover, a serious result about the behavior of a gas in the continuum limit (or incompleteness of the classical gas dynamics) is derived.]

As mentioned at the beginning of this section, we cannot neglect the nonlinear effect for any small Mach number, if it is of the order of the Knudsen number. This is also shown by the numerical analysis of the BKW equation in Sone, Kataoka, Ohwada, Sugimoto & Aoki [1994]. In some infinite domain problem, the variation of the velocity distribution function becomes more and more moderate in the far field or the length scale variation becomes larger and larger there, which means that the effective Knudsen number becomes smaller and smaller. Thus, the nonlinear effect cannot be neglected for any small Mach number, and the linearized Boltzmann equation does not give a uniformly valid solution over the whole domain, which corresponds to the Stokes paradox in the classical gas dynamics (see also Sone, Kataoka, Ohwada, Sugimoto & Aoki [1994], Cercignani [1968], Sone [1978]).

The fluid-dynamic-type equations derived by the S expansion are a series of the Navier–Stokes-type set of equations with “the incompressible Navier–Stokes set of equations” as their leading set, and do not have the inconvenience that the degree of differentiation increases with the process of approximation, which is encountered in the Chapman–Enskog expansion. Further the assumption that the velocity distribution function depends on the space variables only through the five macroscopic variables and their derivatives are unnecessary, and it comes out as a result. The difficulty of the Chapman–Enskog expansion (Chapman & Cowling [1952]) in the boundary-value problem is discussed in Cercignani [1987]. Incidentally, an ill-posed equation is derived from this expansion (see Tamada & Sone [1966], Sone [1968, 1984b]).

### 3.3 Nonlinear problem

Consider the asymptotic behavior for small Knudsen numbers of the cases that are excluded in the previous two sections. The process of analysis is parallel to that in the previous analyses and therefore only its outline and the main results will be given.

### 3.3.1 Gas around a simple boundary

#### A Flow with a finite Mach number

First we try to obtain the solution  $[\partial\phi/\partial x_i = O(\phi)]$  describing the overall behavior of the gas without limiting the size of  $\phi$  in Eq. (1.42) by the power-series expansion in the Knudsen number or  $k$  :

$$\phi = \phi_H = \phi_{H0} + \phi_{H1}k + \cdots . \quad (3.129)$$

Corresponding to the expansion of the velocity distribution function, the macroscopic variables  $\omega, u_i, \tau$ , etc. are also expanded:

$$h_H = h_{H0} + h_{H1}k + \cdots , \quad (3.130)$$

where  $h = \omega, u_i, \tau$ , etc. This series or solution is called Hilbert expansion or Hilbert solution (Hilbert [1924]). The leading term  $\phi_{H0}$  of the expansion is the local Maxwellian characterized by the leading terms  $\omega_{H0}, u_{iH0}$ , and  $\tau_{H0}$  of the five macroscopic variables, i.e., density, flow velocity, and temperature. The variables  $\omega_{H0}, u_{iH0}$ , and  $\tau_{H0}$  are governed by the Euler set of equations:

$$\frac{\partial \tilde{\omega}_{H0} u_{iH0}}{\partial x_i} = 0, \quad (3.131a)$$

$$\tilde{\omega}_{H0} u_{jH0} \frac{\partial u_{iH0}}{\partial x_j} + \frac{1}{2} \frac{\partial P_{H0}}{\partial x_i} = 0, \quad (3.131b)$$

$$\tilde{\omega}_{H0} u_{jH0} \frac{\partial}{\partial x_j} (u_{iH0}^2 + \frac{5}{2} \tilde{\tau}_{H0}) = 0, \quad (3.131c)$$

where

$$\tilde{\omega}_{H0} = 1 + \omega_{H0}, \quad \tilde{\tau}_{H0} = 1 + \tau_{H0}, \quad P_{H0} = \omega_{H0} + \tau_{H0} + \omega_{H0}\tau_{H0}. \quad (3.132)$$

The higher-order macroscopic variables  $\omega_{Hm}, u_{iHm}$ , and  $\tau_{Hm}$  ( $m \geq 1$ ) are governed by inhomogeneous linear Euler-type equations (Hilbert [1924], Grad [1958]).

The correction to make the solution satisfy the kinetic boundary condition (1.24) is carried out in two steps (Grad [1969], Darrozes [1969], Sone, Bardos, Golse & Sugimoto [1999]). First, a solution  $[k^{1/2} n_i \partial\phi/\partial x_i = O(\phi)]$  that varies appreciably over the distance  $k^{1/2}L$  ( $L$  : the characteristic length of the Euler solution) in the direction normal to the boundary is sought. This solution is expanded in a power series of  $k^{1/2}$ , and admits fluid-dynamic-type description as the Hilbert solution.<sup>25</sup> The leading set of equations in the expansion is the viscous boundary-layer set of equations derived from the compressible Navier–Stokes set of equations. The next order set of equations is also of the boundary-layer equation type, and does not contain non-Navier–Stokes terms, contrary to

<sup>25</sup>The Hilbert expansion should be rewritten in  $k^{1/2}$  expansion so as to be connected to the correction solution.

Darroz's claim.<sup>26</sup> At the leading order [ $O(1)$ ], we can make the solution satisfy the kinetic boundary condition by imposing the nonslip boundary condition on the viscous boundary-layer set of equations. At the next order, we have to introduce the Knudsen layer (the solution whose length scale of variation normal to the boundary is of the order of the mean free path) to make the solution satisfy the kinetic boundary condition. The analysis of the Knudsen layer gives the slip boundary condition and the Knudsen-layer correction as in Sections 3.1 and 3.2. At the leading order, they are practically the same as those obtained in the linear or weakly nonlinear analysis.

In a system where the Euler set of equations is the leading set of the fluid-dynamic-type equations, a kinetic transition layer or shock layer, corresponding to a shock wave, may appear in a gas (Grad [1969]). At the zero-th approximation, the solutions of the Euler set of equations across the shock layer are connected by the Rankine–Hugoniot relation for a plane shock wave, and the smooth transition between the two solutions is expressed by the solution of the Boltzmann equation for a plane shock wave to be discussed in Chapter 4.<sup>27</sup>

The introduction of the intermediate layer is due to the fact that a finite Mach number corresponds to a very large Reynolds number when the Knudsen number is very small (see Section 3.1.9).

### **B Gas around a stationary boundary with arbitrary temperature distribution**

Consider a gas around simple boundaries at rest. If the gas region extends to infinity, we assume that neither flow nor pressure gradient is imposed at infinity. We are concerned with the asymptotic behavior for small Knudsen numbers of the system. The steady solution of the Navier–Stokes equations in the classical fluid dynamics for this system is simply as follows: the gas is at rest and the pressure is uniform; the temperature field is determined by the heat-conduction equation (with the thermal conductivity being variable with temperature). Taking this stationary condition into account, we will carry out the asymptotic analysis (Sone & Wakabayashi [1988], Sone, Aoki, Takata, Sugimoto & Bobylev [1996]).

Examining the Hilbert expansion described in Section 3.3.1 A by restricting the local Maxwellian at the leading term of the expansion to a Maxwellian with-

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<sup>26</sup>Our recent analysis (Sone, Bardos, Golse, & Sugimoto [1999]) shows that the non-Navier–Stokes stress and heat flow do not enter in the set of fluid-dynamic-type equations for the macroscopic variables at the leading correction of gas rarefaction ( $k^{1/2}$ -order terms) in spite of Darroz's statement. Thus, the viscous boundary-layer equations are those obtained from the Navier–Stokes equations by  $Re^{-1/2}$  expansion. In the equations, the higher  $k$ -order term of the flow velocity normal to boundary is introduced owing to a strong anisotropic feature of viscous boundary layer. However, it is expressed in terms of lower-order quantities and its boundary value vanishes owing to the exclusion effect of the Knudsen layer being of higher order. Thus, the contribution can be included in the system of the Navier–Stokes equations and the slip conditions consisting of tangential velocity slip due to the shear of flow and temperature jump due to the temperature gradient normal to the boundary.

<sup>27</sup>Except for a weak shock wave, the length scale of variation in its internal structure is of the order of the mean free path, and thus, the analysis by the Boltzmann equation is required.

out flow velocity, we find that the expansion can be carried out consistently.<sup>28</sup> The fluid-dynamic-type equations governing the macroscopic variables at their leading order are given as follows:

$$\frac{\partial \tilde{P}_{H0}}{\partial x_i} = 0, \quad \frac{\partial P_{H1}}{\partial x_i} = 0, \quad (3.133)$$

$$\frac{\partial \tilde{\omega}_{H0} u_{jH1}}{\partial x_j} = 0, \quad (3.134a)$$

$$\begin{aligned} \tilde{\omega}_{H0} u_{jH1} \frac{\partial u_{iH1}}{\partial x_j} = & -\frac{1}{2} \frac{\partial P_{H2}}{\partial x_i} \\ & + \frac{1}{2} \frac{\partial}{\partial x_j} \left[ \hat{\gamma}_1(\tilde{\tau}_{H0}) \tilde{\tau}_{H0}^{1/2} \left( \frac{\partial u_{iH1}}{\partial x_j} + \frac{\partial u_{jH1}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{kH1}}{\partial x_k} \delta_{ij} \right) \right] \\ & - \frac{1}{2 \tilde{P}_{H0}} \frac{\partial}{\partial x_j} \left\{ \hat{\gamma}_7(\tilde{\tau}_{H0}) \left[ \frac{\partial \tilde{\tau}_{H0}}{\partial x_i} \frac{\partial \tilde{\tau}_{H0}}{\partial x_j} - \frac{1}{3} \left( \frac{\partial \tilde{\tau}_{H0}}{\partial x_k} \right)^2 \delta_{ij} \right] \right\} \\ & - \frac{1}{2 \tilde{P}_{H0}} \frac{\partial}{\partial x_j} \left[ \hat{\gamma}_3(\tilde{\tau}_{H0}) \tilde{\tau}_{H0} \left( \frac{\partial^2 \tilde{\tau}_{H0}}{\partial x_i \partial x_j} - \frac{1}{3} \frac{\partial^2 \tilde{\tau}_{H0}}{\partial x_k^2} \delta_{ij} \right) \right], \end{aligned} \quad (3.134b)$$

$$\tilde{\omega}_{H0} u_{jH1} \frac{\partial \tilde{\tau}_{H0}}{\partial x_j} = \frac{1}{2} \frac{\partial}{\partial x_j} \left( \hat{\gamma}_2(\tilde{\tau}_{H0}) \tilde{\tau}_{H0}^{1/2} \frac{\partial \tilde{\tau}_{H0}}{\partial x_j} \right), \quad (3.134c)$$

where

$$\tilde{\omega}_{H0} = 1 + \omega_{H0}, \quad \tilde{\tau}_{H0} = 1 + \tau_{H0}, \quad \tilde{P}_{H0} = 1 + P_{H0} = \tilde{\omega}_{H0} \tilde{\tau}_{H0}, \quad (3.135)$$

and  $\hat{\gamma}_1(\tilde{\tau}_{H0})$ ,  $\hat{\gamma}_2(\tilde{\tau}_{H0})$ ,  $\hat{\gamma}_3(\tilde{\tau}_{H0})$ , and  $\hat{\gamma}_7(\tilde{\tau}_{H0})$  are functions of  $\tilde{\tau}_{H0}$ , which are related to the linearized collision integral and depend on molecular models [ $\hat{\gamma}_1(1) = \gamma_1$ ,  $\hat{\gamma}_2(1) = \gamma_2$ ,  $\hat{\gamma}_3(1) = \gamma_3$  in Sections 3.1 and 3.2], e.g., for a hard-sphere gas and the BKW model,<sup>29</sup>

$$\left. \begin{aligned} \hat{\gamma}_1(\tilde{\tau}_{H0}) &= 1.270042427, \quad \hat{\gamma}_2(\tilde{\tau}_{H0}) = 1.922284066, \\ \hat{\gamma}_3(\tilde{\tau}_{H0}) &= 1.947906335, \quad \hat{\gamma}_7(\tilde{\tau}_{H0}) = 0.189201 \text{ (hard sphere)}, \\ \hat{\gamma}_1(\tilde{\tau}_{H0}) &= \tilde{\tau}_{H0}^{1/2}, \quad \hat{\gamma}_2(\tilde{\tau}_{H0}) = \tilde{\tau}_{H0}^{1/2}, \\ \hat{\gamma}_3(\tilde{\tau}_{H0}) &= \tilde{\tau}_{H0}, \quad \hat{\gamma}_7(\tilde{\tau}_{H0}) = \tilde{\tau}_{H0} \text{ (BKW)}, \end{aligned} \right\} \quad (3.136)$$

In the present expansion,  $u_{iH1}$  is the leading term of the flow velocity, because the case with  $u_{iH0} \equiv 0$  is considered.

<sup>28</sup>From this point of view, the S expansion of Section 3.2.2 corresponds to taking a stationary absolute Maxwellian, i.e., a Maxwell distribution with zero flow velocity and uniform density and temperature as the leading term of the expansion.

<sup>29</sup>For other molecular models, they depend on  $U_0/k_B T_0$  (see the discussion of the dependence of  $\hat{B}$  on the parameter in Section 1.9).

Introducing the Knudsen-layer correction to the Hilbert solution, we can construct the solution satisfying the kinetic boundary condition. By analysis of the Knudsen layer, the boundary condition for the set of fluid-dynamic-type equations (3.134a)–(3.134c) and the Knudsen-layer correction are obtained as follows:

$$\tilde{\tau}_{H0} = \tilde{\tau}_w, \quad \omega_{K0} = \tau_{K0} = 0, \quad (3.137a)$$

$$\begin{bmatrix} u_{iH1} \\ u_{iK1} \end{bmatrix} = -\frac{\tilde{\tau}_{H0}^{1/2}}{\tilde{P}_{H0}} \frac{\partial \tilde{\tau}_{H0}}{\partial x_j} (\delta_{ij} - n_i n_j) \begin{bmatrix} \hat{K}_1 \\ \frac{1}{2} \hat{Y}_1(\tilde{\eta}) \end{bmatrix}, \quad (3.137b)$$

$$\tilde{\eta} = \tilde{P}_{H0} [\tilde{\tau}_w g_M(\tilde{\tau}_w)]^{-1} \eta, \quad \tilde{\tau}_w = 1 + \tau_{w0}, \quad (3.137c)$$

where the quantities with the subscript  $H$  are evaluated on the boundary;  $\hat{K}_1$ ,  $\hat{Y}_1(\tilde{\eta})$ , and  $\tilde{\eta}$  correspond to the slip coefficient  $K_1$  and Knudsen-layer function  $Y_1(\eta)$  of the thermal creep flow and the stretched Knudsen-layer coordinate  $\eta$  defined by Eq. (3.34) in Sections 3.1.4 and 3.1.5, but their relations as well as  $g_M(\tilde{\tau}_w)$  depend on molecular models, kinetic boundary conditions, and  $\tilde{\tau}_w$ ,<sup>30</sup> e.g., for a hard-sphere gas and the BKW model under the diffuse-reflection boundary condition (see Footnote 29),

$$\left. \begin{aligned} \hat{K}_1 = K_1 = -0.6463, \quad \hat{Y}_1(\tilde{\eta}) = Y_1(\tilde{\eta}), \quad g_M(\tilde{\tau}_w) = 1 \quad (\text{hard-sphere}), \\ \hat{K}_1/\tilde{\tau}_w^{1/2} = K_1 = -0.38316, \quad \hat{Y}_1(\tilde{\eta})/\tilde{\tau}_w^{1/2} = Y_1(\tilde{\eta}), \quad g_M(\tilde{\tau}_w)/\tilde{\tau}_w^{1/2} = 1 \quad (\text{BKW}). \end{aligned} \right\} \quad (3.138)$$

The variable  $\tilde{\eta}$  instead of  $\eta$  should be used as the argument of  $Y_1$ , because the temperature variation along the boundary is not so small as in Sections 3.1 and 3.2. In contrast to the case in Section 3.3.1.A, the Knudsen-layer correction is directly made on the restricted Hilbert expansion (HSB expansion).

Equation (3.134b) contains the third-order differential terms. The third-order equation generally requires more boundary conditions than the slip conditions (3.137a) and (3.137b). However, Eq. (3.134b) is of the third order only in its appearance. The following quasi-pressure  $P_{H2}^*$  being introduced:

$$P_{H2}^* = P_{H2} + \frac{2}{3\tilde{P}_{H0}} \frac{\partial}{\partial x_k} \left( \hat{\gamma}_3(\tilde{\tau}_{H0}) \tilde{\tau}_{H0} \frac{\partial \tilde{\tau}_{H0}}{\partial x_k} \right), \quad (3.139)$$

then Eq. (3.134b) is rewritten in the following second-order equation:

$$\begin{aligned} \tilde{\omega}_{H0} u_{jH1} \frac{\partial u_{iH1}}{\partial x_j} = & -\frac{1}{2} \frac{\partial P_{H2}^*}{\partial x_i} \\ & + \frac{1}{2} \frac{\partial}{\partial x_j} \left[ \hat{\gamma}_1(\tilde{\tau}_{H0}) \tilde{\tau}_{H0}^{1/2} \left( \frac{\partial u_{iH1}}{\partial x_j} + \frac{\partial u_{jH1}}{\partial x_i} - \frac{2}{3} \frac{\partial u_{kH1}}{\partial x_k} \delta_{ij} \right) \right] \\ & + \frac{1}{2\tilde{P}_{H0}} \frac{\partial}{\partial x_j} \left\{ \Gamma_7(\tilde{\tau}_{H0}) \left[ \frac{\partial \tilde{\tau}_{H0}}{\partial x_i} \frac{\partial \tilde{\tau}_{H0}}{\partial x_j} - \frac{1}{3} \left( \frac{\partial \tilde{\tau}_{H0}}{\partial x_k} \right)^2 \delta_{ij} \right] \right\}, \end{aligned} \quad (3.140)$$

<sup>30</sup>For some class of molecular models, the dependence of  $\hat{K}_1$  and  $\hat{Y}_1(\tilde{\eta})$  on  $\tilde{\tau}_w$  becomes simple by choosing  $g_M(\tilde{\tau}_w)$  appropriately. For the other,  $g_M(\tilde{\tau}_w)$  may be taken unity.

where

$$\Gamma_7(\tilde{\tau}_{H0}) = \frac{d\hat{\gamma}_3(\tilde{\tau}_{H0})\tilde{\tau}_{H0}}{d\tilde{\tau}_{H0}} - \hat{\gamma}_7(\tilde{\tau}_{H0}). \quad (3.141)$$

For a hard-sphere gas and the BKW model,

$$\Gamma_7(\tilde{\tau}_{H0}) = 1.758705 \quad (\text{hard sphere}), \quad \Gamma_7(\tilde{\tau}_{H0}) = \tilde{\tau}_{H0} \quad (\text{BKW}). \quad (3.142)$$

The  $\partial P_{H2}^*/\partial x_i$  plays the same role as  $\partial P_{H2}/\partial x_i$  (or the pressure term of the incompressible Navier–Stokes equation) and it is eliminated by taking the curl (or rotation) of Eq. (3.140).

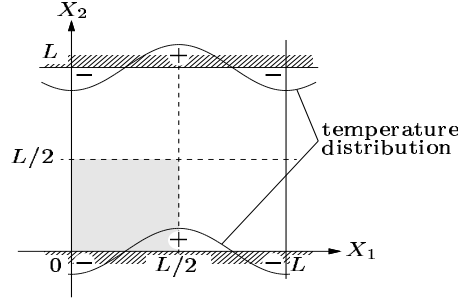
The third and fourth terms on the right hand side of Eq. (3.134b) are due to thermal stress. The thermal stress contribution is not in the form of a gradient of some function, and cannot be incorporated in the pressure term, in contrast to the case in Section 3.2. Therefore, the solution  $u_{iH1} = 0$  is possible under a special temperature field. The condition for  $u_{iH1}$  to be zero is

$$\varepsilon_{ijk} \frac{\partial \tilde{\tau}_{H0}}{\partial x_j} \frac{\partial}{\partial x_k} \left( \frac{\partial \tilde{\tau}_{H0}}{\partial x_l} \right)^2 = 0, \quad (3.143)$$

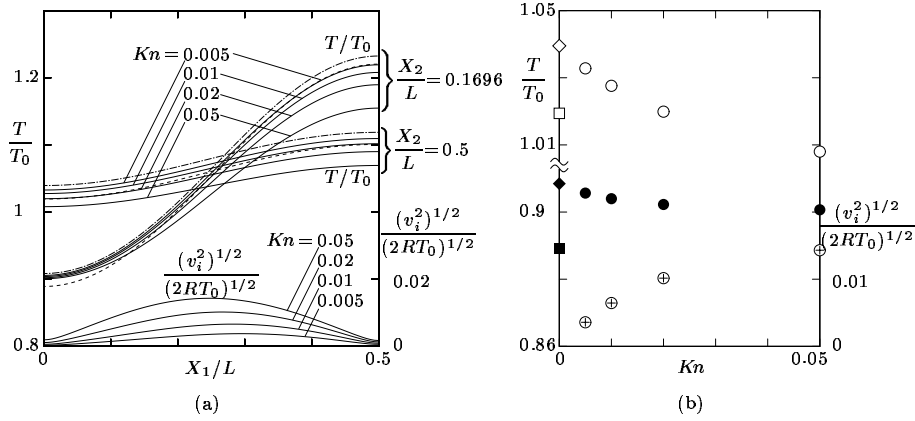
which is obtained by putting  $u_{iH1} = 0$  in Eq. (3.134b). When and only when the distance between isothermal lines or surfaces does not vary along them, the condition is satisfied. Thus, even when the temperature of the boundary is uniform and the thermal creep flow is absent, a flow may be induced in the gas. This flow is called nonlinear thermal stress flow (see also Section 3.4.1.C).

Equations (3.134a)–(3.134c) contain a very important fact. They determine  $\tilde{\tau}_{H0}$ ,  $\tilde{\omega}_{H0}$ ,  $u_{iH1}$ , and  $P_{H2}$  simultaneously. That is, the temperature field  $\tilde{\tau}_{H0}$  in the continuum limit cannot be independent of  $u_{iH1}$ , which is related to gas rarefaction (note:  $u_{iH1}$  is of the order of unity, because the first order rarefaction effect on  $u_{iH}$  is expressed as  $u_{iH1}k$ ). This is strange from the classical gas dynamic point of view. In the world of  $Kn = 0_+$ , the temperature field is affected by something that does not exist in the world, which may be called a ghost from the world of  $Kn \neq 0$ . In the classical gas dynamics, the temperature field of a stationary gas is obtained from the heat-conduction equation, i.e., Eq. (3.134c) with  $u_{iH1} = 0$ . This is allowed only in special cases. As we have seen, the conditions for  $u_{iH1} = 0$  are i) the temperature on each boundary is uniform (the absence of the thermal creep flow) and ii) the distance between isothermal lines or surfaces does not change along them. These are satisfied only in the very special case where each of the boundaries is of uniform temperature and of a uniform mean curvature, for example, in a gas between two parallel plane walls, between coaxial cylinders, or between concentric spheres with uniform temperature on each boundary. Thus, the heat-conduction equation is generally inappropriate in describing the temperature field of a stationary gas in the continuum limit.

In order to understand the situation more clearly, we take a simple example and compare three kinds of solutions: the solution of the heat-conduction equation, the solution of the system derived by the asymptotic analysis [Eqs. (3.134a–c) and Eqs. (3.137a, b)], and the numerical solutions of the Boltzmann equation

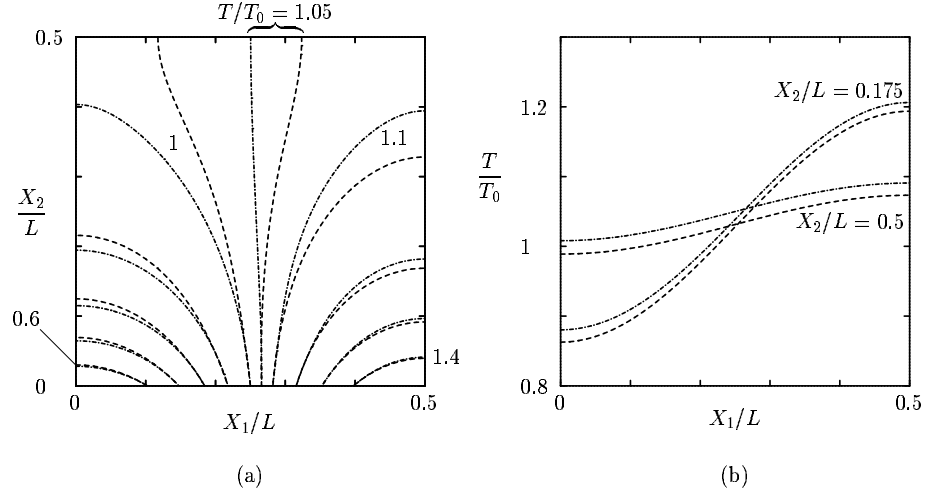


**Figure 3.3.** Geometry. In this example, the behavior of a gas between two parallel plane walls with the periodic temperature distribution  $T_w = T_0(1 - \Delta\tau_w \cos 2\pi X_1/L)$  is considered.



**Figure 3.4.** Comparison of the three kinds of solutions: the solution of the heat-conduction equation, the solution of the system derived by the asymptotic analysis, and the numerical solutions of the Boltzmann equation for various small Knudsen numbers, in the problem (with  $\Delta\tau_w = 0.5$ ) of Fig. 3.3. (a) Temperature ( $T$ ) distributions along  $X_2/L = 0.1696$  and  $X_2/L = 0.5$  and flow-speed  $[(v_i^2)^{1/2}]$  distribution along  $X_2/L = 0.0227$ . (b) Temperature at  $(X_1/L, X_2/L) = (0, 0.1696)$  and  $(0, 0.5)$  and flow speed at  $(X_1/L, X_2/L) = (0.25, 0.0227)$  vs  $Kn$ . In (a), —: BKW equation, - - - : asymptotic theory, and - - - : heat-conduction equation. In (b), white symbols ( $\circ$ ,  $\diamond$ ,  $\square$ ):  $T/T_0$  at  $(X_1/L, X_2/L) = (0, 0.5)$ , black symbols ( $\bullet$ ,  $\blacklozenge$ ,  $\blacksquare$ ):  $T/T_0$  at  $(X_1/L, X_2/L) = (0, 0.1696)$ , and  $\oplus$ :  $(v_i^2)^{1/2}/(2RT_0)^{1/2}$  at  $(X_1/L, X_2/L) = (0.25, 0.0227)$ ; circles ( $\circ$ ,  $\bullet$ ,  $\oplus$ ): BKW equation, diamonds ( $\diamond$ ,  $\blacklozenge$ ): asymptotic theory, and squares ( $\square$ ,  $\blacksquare$ ): heat-conduction equation. Note the difference of the ordinate of  $T/T_0$  from that of  $(v_i^2)^{1/2}/(2RT_0)^{1/2}$  in both figures. The solution of the kinetic equation converges to the asymptotic solution.





**Figure 3.5.** Comparison of the solutions of the asymptotic theory and heat-conduction equation for the hard-sphere molecules in the problem (with  $\Delta\tau_w = 0.5$ ) of Fig. 3.3. (a) The isothermal lines  $T/T_0 = 0.6 + 0.1m$  ( $m = 0, 1, \dots, 8$ ) and 1.05. (b) The temperature distributions along  $X_2/L = 0.175$  and  $X_2/L = 0.5$ . Here,  $\cdots$ : asymptotic theory, and  $---$ : heat-conduction equation.

for various small Knudsen numbers. The example considered here is a gas between two parallel plane walls at  $X_2 = 0$  and  $X_2 = L$ ; both walls are at rest and have a common temperature distribution  $T_w = T_0(1 - \Delta\tau_w \cos 2\pi X_1/L)$ , where  $T_0$  and  $\Delta\tau_w$  are constants (Fig. 3.3). We look for the periodic solution with period  $L$  with respect to  $X_1$ . In Fig. 3.4, the three kinds of solutions for  $\Delta\tau_w = 0.5$  are compared, where the BKW equation (and the corresponding asymptotic and heat-conduction equations) is used for simplicity of numerical computation of the kinetic equation, because the principal features of the previous discussion are the same for the BKW equation and the standard Boltzmann equation. The temperature distributions along  $X_2/L = 0.1696$  and  $X_2/L = 0.5$ , and the flow speed distribution along  $X_2/L = 0.0227$  are shown in Fig. 3.4 (a); the temperature at  $(X_1/L, X_2/L) = (0, 0.1696)$  and  $(0, 0.5)$  and the flow speed at  $(0.25, 0.0227)$  versus  $Kn$  are shown in Fig. 3.4 (b). The Knudsen number  $Kn$  here is defined by the mean free path of the average density of the gas in the domain and the channel width  $L$ . Clearly from Figs. 3.4 (a) and (b), as  $Kn$  tends to zero, the temperature field of the kinetic equation approaches that of the system of the asymptotic theory and not that of the heat-conduction equation, and the flow vanishes. In Figs. 3.5 (a) and (b), a comparison of the solution of the asymptotic system and that of the heat-conduction equation for a hard-sphere molecular gas is shown for the same problem.

The possible cause of the velocity  $u_{iH1}$  is not only the temperature field but also the motion of the boundary of the order of  $Kn$ . The extension of the system (3.133)–(3.134c) and (3.137a,b) to the case where the boundary is moving with

velocity  $(2RT_0)^{1/2}u_{wi1}k$  ( $u_{wi1}n_i = 0$ ) is simple; we have only to replace  $u_{iH1}$  in Eq. (3.137b) by  $u_{iH1} - u_{wi1}$  (Sone, Aoki, Takata, Sugimoto & Bobylev [1996]). This poses a strange problem. The solution of the temperature field at  $Kn = 0_+$  is indefinite because we cannot specify  $u_{wi1}$  in the world  $Kn = 0_+$ , in contrast to the case of the temperature-induced flow, where  $u_{iH1}$  (thus  $\tilde{\tau}_{H0}$ ) is determined uniquely by the boundary data  $\tilde{\tau}_w$ . In other words, a very small motion of the boundary (of the order of  $Kn$ ) affects the temperature field considerably (of the order of unity) for very small Knudsen numbers.

Thus, the classical gas dynamics (or continuum gas dynamics) is incomplete to describe the behavior of a gas even in the continuum limit (see also Sone, Aoki, Takata, Sugimoto & Bobylev [1996] and Sone [1997, 2000a]). This important fact will be discussed again in Section 3.5 from a different view point.

### 3.3.2 Flow with strong evaporation or condensation

Consider the case where the boundary is a condensed phase of the gas and strong evaporation or condensation is taking place. The “strong” simply means that we allow a high speed evaporation or condensation with Mach number of the order of unity, in contrast to the cases in Sections 3.1 and 3.2. The solution is obtained in the form of the standard Hilbert solution and its Knudsen-layer correction (Aoki & Sone [1991]).

The fluid-dynamic-type equations that come first in the expansion are the Euler set of equations:

$$\frac{\partial \rho v_i}{\partial X_i} = 0, \quad (3.144a)$$

$$\rho v_j \frac{\partial v_i}{\partial X_j} + \frac{\partial p}{\partial X_i} = 0, \quad (3.144b)$$

$$v_j \frac{\partial}{\partial X_j} \left( \frac{5}{2} RT + \frac{1}{2} v_i^2 \right) = 0, \quad (3.144c)$$

with

$$p = R\rho T, \quad (3.145)$$

which are the dimensional form of Eqs. (3.131a)–(3.132) with the subscripts  $H$  and 0 being omitted.

The Knudsen-layer correction of the leading order, which appears at the zero order of the Knudsen number, is obtained from the solution of the half-space evaporation-condensation problem. In contrast to the case of a simple boundary, this half space problem is a nonlinear problem (a problem of the one-dimensional Boltzmann equation without linearization). It is extensively studied numerically on the basis of the BKW equation and the complete-condensation condition [Eqs. (1.26a) and (1.26b) with  $\alpha_c = 1$ ] in Sone, Aoki & Yamashita [1986], Sone, Aoki, Sugimoto & Yamada [1988], Sone & Sugimoto [1990], Aoki, Sone & Yamada [1990], and Aoki, Nishino, Sone & Sugimoto [1991]. According to them, the boundary condition for the Euler set of equations is given as follows.

**Table 3.3.** The functions  $h_1(M_n)$  and  $h_2(M_n)$ .

$M_n$	$h_1$	$h_2$	$M_n$	$h_1$	$h_2$	$M_n$	$h_1$	$h_2$
0.0000	1.0000	1.0000	0.4000	0.4900	0.8470	0.8000	0.2695	0.7088
0.04999	0.9083	0.9798	0.4400	0.4593	0.8326	0.8400	0.2553	0.6956
0.07998	0.8582	0.9679	0.4800	0.4310	0.8184	0.8800	0.2420	0.6824
0.1200	0.7966	0.9521	0.5200	0.4050	0.8043	0.9200	0.2297	0.6693
0.1600	0.7404	0.9365	0.5600	0.3809	0.7904	0.9600	0.2182	0.6563
0.2000	0.6891	0.9212	0.6000	0.3586	0.7765	0.9700	0.2155	0.6530
0.2400	0.6421	0.9060	0.6400	0.3380	0.7628	0.9800	0.2128	0.6498
0.2800	0.5991	0.8910	0.6800	0.3189	0.7492	0.9900	0.2101	0.6466
0.3200	0.5596	0.8761	0.7200	0.3012	0.7356	1.0000	0.2075	0.6434
0.3600	0.5233	0.8615	0.7600	0.2848	0.7222			

In the following formulas, the notation

$$M_n = |v_i n_i| (5RT/3)^{-1/2}, \quad M_t = |v_i - v_j n_j n_i - v_{wi}| (5RT/3)^{-1/2}, \quad (3.146)$$

is used.

**(a) In the case of evaporation** ( $v_i n_i > 0$ , or  $p/p_w < 1$ )

$$M_n \leq 1, \quad p/p_w = h_1(M_n), \quad T/T_w = h_2(M_n), \quad M_t = 0, \quad (3.147)$$

where  $p_w$  is the saturation gas pressure at the temperature  $T_w$  of the condensed phase. The functions  $h_1(M_n)$  and  $h_2(M_n)$  for the BKW equation with the complete-condensation condition are tabulated in Table 3.3.

**(b) In the case of condensation** ( $v_i n_i < 0$ , or  $p/p_w > 1$ )

$$p/p_w = F_s(M_n, M_t, T/T_w), \quad \text{when } M_n < 1, \quad (3.148a)$$

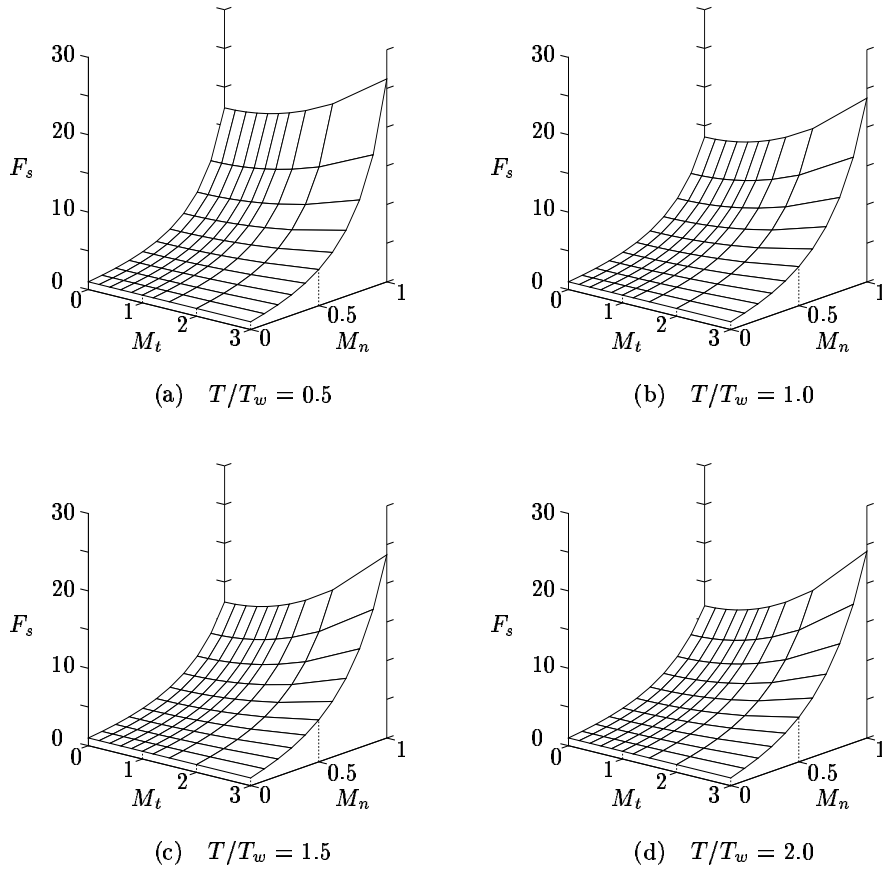
$$p/p_w > F_b(M_n, M_t, T/T_w), \quad \text{when } M_n > 1, \quad (3.148b)$$

where the relation  $\geq$ , instead of  $>$ , holds at  $M_n = 1$  in the second formula exceptionally. Examples of the functions  $F_s$  and  $F_b$  for the BKW equation with the complete-condensation condition are given in Figs. 3.6 and 3.7.<sup>31</sup>

These boundary conditions for evaporation and condensation for the complete-condensation condition can be generalized by a simple transformation to the more general mixed type condition [Eqs. (1.26a) and (1.26b)] with  $\alpha = 1$  and  $\alpha_c$  being arbitrary (see Aoki & Sone [1991], Sone & Sugimoto [1990], Sone, Aoki & Doi [1992]). This generalization applies to the standard Boltzmann equation (not limited to the BKW equation).

The boundary condition on an interface shows qualitatively different characters depending on evaporation, subsonic condensation, or supersonic condensation. In the four dimensional space  $(M_n, M_t, p/p_w, T/T_w)$ , the boundary data

<sup>31</sup>Recent direct simulation Monte Carlo computations of the subsonic condensation for hard-sphere molecules for the cases  $M_t = 0$  and  $T/T_w = 0.5, 1, 2$  (Sone & Sasaki [unpublished]) show that the relative differences of  $p/p_w$  from the BKW results are less than 1% except in the case  $0.9 \leq M_n \leq 1$  at  $T/T_w = 0.5$ , where the differences are bounded by 5%.



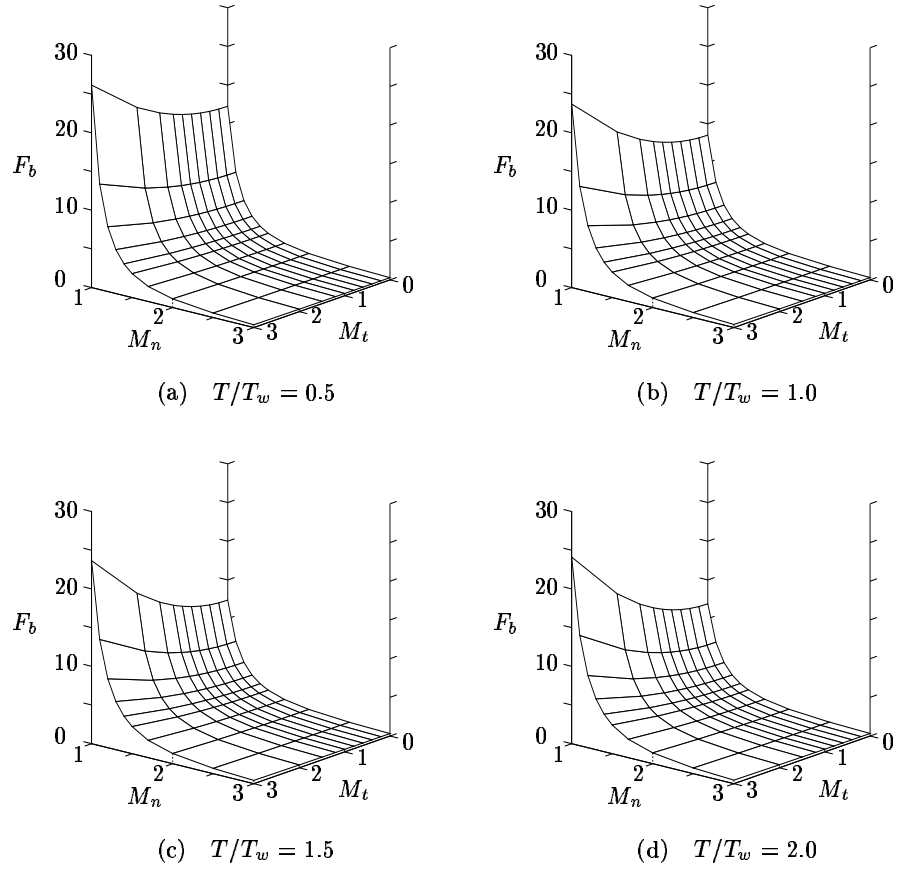
**Figure 3.6.**  $F_s(M_n, M_t, T/T_w)$ .

are given on a hyper-curve for evaporation, on a hyper-surface for subsonic condensation, and in a domain for supersonic condensation. The analytical structure of this transition is clarified in Sone [1978] (from evaporation to condensation) and Sone, Golse, Ohwada & Doi [1998] (from subsonic condensation to supersonic condensation). See more comprehensive discussion in Sone [2000b].

As in the case of Section 3.3.1.A, a shock layer may appear in a flow of the present problem.

### 3.4 Application

The asymptotic theory developed in this chapter has wide application. Here a few examples, which are particular to a rarefied gas or are found by kinetic theory consideration, are discussed with the aid of the asymptotic theory.

Figure 3.7.  $F_b(M_n, M_t, T/T_w)$ .

### 3.4.1 Flows induced by temperature field

In the framework of the classical gas dynamics, no steady flow is induced in a gas without external force, such as gravity, by the effect of temperature field. In a rarefied gas, on the other hand, the temperature field of a gas (often in combination with a solid boundary) plays an important role on inducing a steady flow. In this section, we will point out the flows induced by a temperature field in a slightly rarefied gas on the basis of the asymptotic theory, and explain their physical mechanisms. More comprehensive discussion on temperature-induced flows will be given in Chapter 4 (see also Sone [2000a]).

#### A Thermal creep flow

On a simple boundary in a slightly rarefied gas, the slip condition (3.43a) imposes a flow, tangential to the boundary and proportional to the temperature gradient of the boundary, as the boundary condition of the Stokes set of equa-

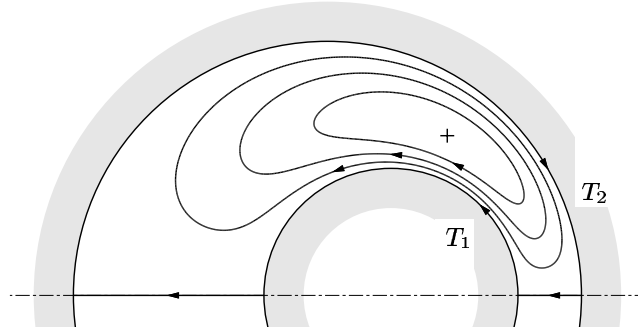
tions. Thus, a global flow is induced in a gas around a body with nonuniform temperature. The flow, particular to a rarefied gas, has been known for a long time as thermal creep flow (Kennard [1938]), but its rigorous formulation and accurate analysis on the basis of the Boltzmann equation are relatively new (Sone [1966, 1970], Ohwada, Sone & Aoki [1989]). The physical mechanism of this flow is given as follows.

Consider a gas at rest over a plane wall with a temperature gradient along it. Take a small area  $dS$  on the wall and estimate the momentum transferred to  $dS$  from the gas. The molecules impinging on  $dS$  come from various directions directly (or without molecular collisions) over a distance of the order of the mean free path, keeping the property of their origins. The average speed of molecules arriving from the hotter region is larger than that of molecules from the colder region. Thus the momentum transferred to  $dS$  by the molecules impinging on it has a component in the direction opposite to the temperature gradient on the wall. (A little more precisely, the number of molecules in the hotter region is less than that in the colder region, and the mass fluxes from the two regions balance in a gas at rest. The momentum transfer, to which the velocity of a molecule contributes more magnified, has a component in the direction opposite to the temperature gradient.) The contribution of the molecules leaving the wall to the tangential component of the momentum transfer is nothing in the case of the diffuse reflection. (Generally except in the specular reflection, the velocity distribution function of the molecules leaving the wall, characterized by the wall condition, is qualitatively different from that of the impinging molecules, formed by collision of surrounding gas molecules. Thus the two contributions are generally different.) Thus, a momentum in the opposite direction of the temperature gradient is transferred to  $dS$  from the gas. As its reaction, the gas is subject to a force in the direction of the temperature gradient, and a flow is induced in that direction. In a gas in motion, on the other hand, a momentum in the direction of motion is transferred to the wall or  $dS$ . Thus, a steady flow is established when the two contributions of momentum transfer balance.

The thermal creep flow is demonstrated and its relative magnitude (with respect to the temperature gradient of the boundary and the pressure of the gas) is measured by simple experiments in Sone [1991b] and Sone, Sawada & Hirano [1994].

## B Thermal stress slip flow

Consider a gas between two noncoaxial circular cylinders with different uniform temperatures  $T_1$  and  $T_2$ . Then, no thermal creep flow is induced. There is a temperature gradient on each cylinder normal to it, and it is nonuniform, i.e.,  $(\partial G_{iG0}/\partial x_j)n_i t_j [= -(\partial^2 \tau_{G0}/\partial x_i \partial x_j)n_i t_j] \neq 0$ . According to (3.44a), therefore, a global flow as shown in Fig. 3.8 is induced between the cylinders (Sone [1971, 1972, 1974], Sone & Tanaka [1980]). The flow generally induced owing to the term  $(\partial G_{iG0}/\partial x_j)n_i t_j$  in the second-order slip boundary condition (3.44a) is called thermal stress slip flow, whose origin is explained in the next paragraph. One may naturally think that the flow is a kind of the thermal creep flow due to



**Figure 3.8.** Thermal stress slip flow. The direction of flow is shown by arrows on streamlines when the temperature  $T_2$  of the outer cylinder is higher than the temperature  $T_1$  of the inner cylinder.

the modified temperature field by the effect of the first-order temperature jump (3.43c); but this is the opposite direction to the thermal stress slip flow.

As is seen in Section 3.1.3, the thermal stress  $\partial^2 \tau_{G0} / \partial x_i \partial x_j$  appears with advance of approximation [see Eq. (3.24)]. The average speed of molecules that arrive at a point under consideration depends on the direction from which they come directly (or without molecular collision) over a distance of the order of the mean free path, when the temperature gradient is nonuniform. This introduces a momentum flux resulting in the thermal stress. (The form of stress in a two-dimensional case can easily be understood by considering the representative molecules from four different directions.) The stress in the linearized problem does not have any contribution to the Stokes set of equations. Take a small cube in the gas. Then, the thermal stress over the surface of the cube balances by itself. One of the squares of the cubic is assumed here to lie on an isothermal surface and to be replaced by a solid wall where the diffuse reflection takes place. Then, the tangential momentum flux through the surface is reduced by half, and the balance of the momentum flux (or thermal stress) over the cubic is violated. Thus, the thermal stress slip flow is induced. Obviously from the physical explanation, a solid boundary (or the qualitative difference of the velocity distribution functions of the molecules impinging on the boundary and those leaving there) plays an essential role on the thermal stress slip flow as well as the thermal creep flow.

### C Nonlinear thermal stress flow

Another thermal-stress induced flow is proposed in Kogan, Galkin & Friedlander [1976]. When temperature gradient in a gas is so large that its nonlinear effect is not negligible, as explained in Section 3.3.1.B, the thermal stress does not balance by itself and remains in the conservation equation of momentum. This causes a flow of the first order of the Knudsen number when the distance between isothermal lines or surfaces varies along them. In this flow, in contrast to the

thermal creep flow and thermal stress slip flow, a solid boundary plays only an indirect role to form such a temperature field.

### D Flow induced around the edge of a heated plate

In the asymptotic analysis developed in this chapter, we assumed that the shape of the boundary was smooth, i.e., the Knudsen number based on the radius of curvature of the boundary was small. Thus the case where the boundary has a sharp edge is excluded. Recently it is found numerically and experimentally that a flow is induced around the edge of a heated plate (Aoki, Sone & Masukawa [1995], Sone & Yoshimoto [1997]). According to a crude estimate in Sone & Yoshimoto [1997], the flow speed is of the order of  $\sqrt{Kn}$ . We will discuss more about the flow in Chapter 4 (see also Sone [2000a]).

### 3.4.2 Negative temperature gradient phenomenon

The boundary condition for the fluid-dynamic-type equations on an interface of a gas and its condensed phase is derived in Sections 3.1.5, 3.2.3, and 3.3.2. The formula for the leading term of the expansion is the boundary condition that is to be applied to the fluid-dynamic equations in the continuum gas dynamics. (Strangely, the condition on the interface seems to be not seriously studied among classical fluid-dynamicists in spite of its importance.) An interesting phenomenon, called negative temperature gradient phenomenon, was found on the basis of the boundary condition on the interface derived by kinetic theory.

Consider a gas between its two parallel condensed phases with different temperatures  $T_1$  and  $T_2$  ( $T_2 > T_1$ ). The saturation gas pressure  $p_s$  is a fairly rapidly increasing function of temperature [ $p_w^{(2)} = p_s(T_2) > p_s(T_1) = p_w^{(1)}$ ], which is determined by the Clausius–Clapeyron relation (Reif [1965], Landau & Lifschitz [1963]). Assuming that the temperature difference of the condensed phases is small [ $(T_2 - T_1)/T_1 \ll 1$ ], we discuss the problem on the basis of the linear theory in Section 3.1, and use the notation there. Let  $(\tau^{(1)}, P_w^{(1)})$  and  $(\tau^{(2)}, P_w^{(2)})$  be, respectively,  $(\tau_w, P_w)$  corresponding to temperatures  $T_1$  and  $T_2$ . Let the  $x_1$  coordinate be normal to the condensed phase and let the condensed phase at temperature  $T_1$  and that at  $T_2$  be set, respectively, at  $x_1 = 0$  and  $x_1 = 1$ . According to Eqs. (3.12) and (3.13a),  $P_{G0} = \text{const}$ , and  $u_{1G0} = \text{const}$ , because the quantities may be assumed to depend only on  $x_1$ . From the boundary condition (3.57c) at each condensed phase, these constants are determined as

$$P_{G0} = (P_w^{(1)} + P_w^{(2)})/2, \quad u_{1G0} = (P_w^{(2)} - P_w^{(1)})/2C_4^*. \quad (3.149)$$

Here,  $u_{1G0}$  is negative (or evaporation at  $x_1 = 1$ , and condensation at  $x_1 = 0$ ), because  $C_4^*$  is negative.

Corresponding to condensation at  $x_1 = 0$  and evaporation at  $x_1 = 1$ , there are a temperature jump, i.e.,  $\tau_{G0} - \tau^{(1)} = (d_4^*/2C_4^*)(P_w^{(2)} - P_w^{(1)})$ , at  $x_1 = 0$  and a drop, i.e.,  $\tau_{G0} - \tau^{(2)} = -(d_4^*/2C_4^*)(P_w^{(2)} - P_w^{(1)})$ , at  $x_1 = 1$ . With these conditions at the condensed phases,  $\tau_{G0}$ , the solution of Eq. (3.13c) with



$m = 0$ , obviously varies linearly in  $x_1$  from  $\tau^{(1)} + (d_4^*/2C_4^*)(P_w^{(2)} - P_w^{(1)})$  to  $\tau^{(2)} - (d_4^*/2C_4^*)(P_w^{(2)} - P_w^{(1)})$  as

$$\tau_{G0} = \tau^{(1)} + \frac{d_4^*}{2C_4^*}(P_w^{(2)} - P_w^{(1)}) + \left( (\tau^{(2)} - \tau^{(1)}) - \frac{d_4^*}{C_4^*}(P_w^{(2)} - P_w^{(1)}) \right) x_1. \quad (3.150)$$

If the condition  $(P_w^{(2)} - P_w^{(1)})/(\tau^{(2)} - \tau^{(1)}) > C_4^*/d_4^*$  ( $= 4.6992$  for a hard-sphere molecular gas,  $= 4.7723$  for the BKW model) or

$$\frac{T_1}{p_s(T_1)} \left( \frac{p_s(T_2) - p_s(T_1)}{T_2 - T_1} \right) > \frac{C_4^*}{d_4^*}, \quad (3.151)$$

is fulfilled, which is so for many kinds of gases, then  $\tau_{G0}$  at  $x_1 = 1$  is smaller than  $\tau_{G0}$  at  $x_1 = 0$ , and therefore the temperature gradient in the gas is in the opposite to that imposed between the two condensed phase. This phenomenon, first pointed out by Pao [1971], is called negative temperature gradient phenomenon, and is also studied from the point of the irreversible thermodynamics (Hermans & Beenakker [1986]). As is obviously from the above analysis, the negative temperature phenomenon is due to the temperature jump condition (3.57c). The physical reason that the temperature drop occurs on an evaporating condensed phase (or a jump on a condensing condensed phase) is discussed in Sone & Onishi [1978].

When the temperature difference between the condensed phases is not small, the problem can be treated equally easily with the aid of Sone [1978], Onishi & Sone [1979], Sone & Aoki [1987], or Aoki & Sone [1991]. Then, the temperature field is affected by convection of gas flow, but the negative temperature gradient character remains unchanged (The solution for the BKW equation, which is only a simple application of Sone [1978] or Onishi & Sone [1979], is given in Aoki & Cercignani [1983]). When the Mach number of the flow is of the order of unity, the convection effect is so strong that the temperature field becomes uniform except in the Knudsen layer (Aoki & Sone [1991]).

This two-surface problem, studied in more detail including the case of arbitrary Knudsen number (Sone & Onishi [1978], Sone, Ohwada & Aoki [1991], Aoki & Masukawa [1994]) or experimentally (Shanker & Deshpande [1990]), will be discussed again in Chapter 5.

### 3.5 Essential defect of the continuum gas dynamics

In Section 3.3.1.B, we pointed out that the classical gas dynamics (continuum gas dynamics) had an essential defect in describing the behavior of the temperature field of a stationary gas in the continuum limit ( $Kn = 0_+$ ). Here we discuss the problem on the basis of the Navier–Stokes set of equations.

The energy equation of the Navier–Stokes set of equations is given as follows:

$$\frac{5}{2}\rho v_i \frac{\partial RT}{\partial X_i} = \cdots + \frac{\partial}{\partial X_i} \lambda \frac{\partial T}{\partial X_i}, \quad (3.152)$$

where only the terms essential to the discussion are shown, and the thermal conductivity  $\lambda$  is generally a function of the temperature of the gas. In discussing the temperature field of a gas at rest, one simply put  $v_i = 0$  and solve the heat-conduction equation  $\partial(\lambda \partial T / \partial X_i) / \partial X_i = 0$ . However, we have to examine the equation (3.152) a little more carefully. The thermal conductivity depends on the molecule constituting the gas. According to elementary kinetic theory or rigorous analysis of the Boltzmann equation [e.g., Kennard [1938] or Eq. (3.72)], the thermal conductivity divided by the density of the gas is proportional to the mean free path, that is,

$$\lambda / \rho = g(T) (2RT)^{1/2} R \ell,$$

where  $g(T)$  is a nondimensional function of  $T$  [e.g.,  $g(T) = 2.1295$  for a hard-sphere gas]. Thus the conduction term (the last term on the right hand side) of the energy equation divided by the density is of the order of the mean free path. Then, for a gas flow of the order of the mean free path, which is neglected in the classical gas dynamics, its convection term (its left hand side) is of the same order as the conduction term and cannot be neglected. In other words, the order of the convection term is  $\rho V R T / L$  and that of the conduction term is  $\rho (RT)^{3/2} \ell / L^2$ , where  $V$  and  $L$  are, respectively, the characteristic flow speed of the gas and the length scale of variation of the temperature of the gas. Thus, the two terms are comparable when  $V$  is the order of  $(RT)^{1/2} \ell / L$  (or the Mach number of the flow is of the order of the Knudsen number). As a flow of this magnitude, we know the thermal creep flow or nonlinear thermal stress flow, which is inevitable except in special cases (Sections 3.3.1 B and 3.4.1). Thus, the heat-conduction equation is inappropriate in describing the temperature field of a stationary gas in the continuum limit. See also Sone [2000a].

This kind of ghost effect in the sense that something that does not exist in the world of a gas in the continuum limit has a finite effect in this world is also found in the cylindrical Couette flow with evaporation or condensation on the cylinders in Sone, Takata & Sugimoto [1996] (see also Sone [1997]). In this example the effect appears also in the flow-velocity field.

As we have seen, the continuum gas dynamics is incomplete. In order to analyze the ghost effect and to find the correct behavior of a gas in the continuum limit ( $Kn = 0_+$ ), we have to rely on molecular gas dynamics or kinetic theory. This is a new important role of molecular gas dynamics.



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